SUFFICIENT CONDITIONS FOR NASH IMPLEMENTATION

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#	Author(s)	Title	#	Author(s)	Title
1		Workshop Summaries from the September 1982 workshop on Statistical Mechanics, Dynamical Systems and Turbulence	20	George R. Sell	Linearization and Global Dynamics
2	Raphael De laLlave	A Simple Proof of C. Siegel's Center Theorem	21	P. Constantin C. Foias	Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 2D Navier-Stokes Equations
3	H. Simpson, S. Spector	On Copositive Matrices and Strong Ellipticity for Isotropic Elastic Materials	22	Milan Miklavcic	Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator
4	George R. Sell	Vector Fields in the Vicinity of a Compact Invariant Manifold	23	P. Collet, H. Epstein	Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties
5	Milan Miklavcic	Non-linear Stability of Asymptotic Suction	24	J.E. Dunn, J. Serrin	On the Thermomechanics of Interstitial working
6	Hans Weinberger	A Simple System with a Continuum of Stable Inhomogeneous Steady States	25	Scott J. Spector	On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension
7	Bau-Sen Du	Period 3 Bifurcation for the Logistic Mapping	26	W.A. Coppel	Maps on an Interval
8	Hans Weinberger	Optimal Numerical Approximation of a Linear Operator	27	James Kirkwood	Phase Transitions in the Ising Model with Traverse Field
9	L.R. Angel, D.F. Evans, B. Ninham	Three Component Ionic Microemulsions	28	Luis Magalhães	The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different
10	D.F. Evans, D. Mitchell, S. Mukherjee, B. Ninham	Surfactant Diffusion; New Results and Interpretations	29	Charles Tresser	Homoclinic Orbits for Flow in R^3
11	Leif Arkeryd	A Remark about the Final Aperiodic Regime for Maps on the Interval	30	Charles Tresser	About some Theorems by L.P. Sil'nikov
12	Luis Magalhães	Manifolds of Global Solutions of Functional Differential Equations	31	Michael Alzenmann	On the renormalized Coupling Constant and the Susceptibility in ϕ_4 Field theory and the Ising Model in Four Dimensions
13	Kenneth Meyer	Tori in Resonance	32	Gene Wayne	The KAM Theory of Systems with Short Range Interactions I
14	C. Eugene Wayne	Surface Models with Nonlocal Potentials: Upper Bounds	33	M. Slemrod J.E. Marsden	Temporal and Spatial Chaos in a Van der Waals Fluid due to Periodic Thermal Fluctuations
15	Kathy Pericak-Spector	On Stability and Uniqueness of Fluid Flow Through a Rigid Porous Medium	34	J. Kirkwood, C.E. Wayne	Percolation in Continuous Systems
16	George R. Sell	Smooth Linearization Near a Fixed Point	35	Luis Magalhães	Invariant Manifolds for Functional Differential Equations Close to Ordinary Differential Equations
17	David Wolikind	A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification	36	C. Eugene Wayne	The KAM Theory of Systems with Short Range Interactions II
18	Pierre Collet	Local C^∞ Conjugacy on the Julia Set for some Holomorphic Perturbations of $z \rightarrow z^2$	37	Jean De Canniere	Passive Quasi-Free States of the Noninteracting Fermi Gas
19	Henry C. Simpson Scott J. Spector	On the Modified Bessel Functions of the First Kind On Barrelling for a Material in Finite Elasticity	38	Elias C. Aifantis	Maxwell and van der Waals Revisited
			39	Elias C. Aifantis	On the Mechanics of Modulated Structures

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ABSTRACT

Given a performance standard which satisfies monotonicity and no veto power, Maskin constructed a game which he claimed would implement this performance standard in Nash equilibrium. A counter-example to this construction is discussed in this paper: a performance standard which satisfies both monotonicity and no veto power is presented which is not implemented by the Nash equilibria of Maskin's game. The flaw in Maskin's argument is described, and a new proof of his result is also presented.

What properties characterize those performance standards (or social choice correspondences) which can be implemented by the Nash equilibria of some game? In 1977, Maskin [3, Thm. 5] stated necessary and (slightly stronger) sufficient conditions which almost completely answer this question when there are at least three agents. He proved that every performance standard which could be implemented in Nash equilibrium by some game must be monotonic, and he also asserted that any monotonic performance standard which satisfies no veto power could be implemented in Nash equilibrium by some game ("monotonicity" and "no veto power" will be defined in Section 2). His proof that monotonicity and no veto power are sufficient was constructive: given a performance standard which has these properties, Maskin defined a game which he claimed would implement the performance standard in Nash equilibrium.

Maskin's construction has several limitations, however. First, he implicitly assumes in his construction that the performance standard is surjective, i.e., every social alternative is assigned by the performance standard to at least one value of the environment.¹ This seems like an innocuous hypothesis, but the designation of a particular set as the set of social alternatives plays a key role in determining whether or not the performance standard is monotonic. A performance standard can be monotonic when some large set is considered as the set of social alternatives, but it might not be monotonic when some smaller set which contains the range of the performance standard is used instead. Secondly, Maskin demonstrates his construction only in the case where the set of social alternatives is finite. The extension of this result to more interesting cases in nontrivial.

In Williams [7], Maskin's result is shown to be a special case of a more general procedure for using "resource allocation processes" [6, p. 170] which "realize" (or compute) a given performance standard to construct games which implement the performance standard in Nash equilibrium. The proofs of the

¹ Otherwise, the game which he constructs would clearly not satisfy hypothesis (5) of his Theorem 4 [3, p. 19]. This hypothesis can be found below in statement (5) of this paper.

results in that paper do not depend directly upon the "size" of the set of social alternatives.

In this same paper, a flaw in Maskin's construction is pointed out: the game which Maskin defines may not implement the given performance standard in Nash equilibrium unless an additional hypothesis is made about the performance standard. The different hypotheses which are suggested in [7] to insure that Maskin's game will work as intended are very general: the cases where Maskin's construction would not work are truly exceptional.

There are two main topics in this paper. First, we will present an example to illustrate that Maskin's construction may not be successful. This example is intended to clarify the more theoretical discussion in [7]. As mentioned above, that paper handles the difficulty which can arise in Maskin's construction by imposing an additional hypothesis on the performance standard. Our second goal in this paper is to show that an additional hypothesis is unnecessary. Essentially, we will prove Maskin's result as it was originally stated. By referring to the construction procedure which is outlined in [7], however, we will not need to make an implicit assumption about the size of the set of social alternatives.

In our next section, we will first summarize the notation and the terminology that we will use in this paper. Following this, we will state exactly what Maskin attempted to prove in Theorem 5 of [3], and we will also state a corresponding theorem which follows from the results of Williams [7]. Williams' result uses hypotheses which are not stated in Maskin's result; the need for these hypothesis will be explained. In Section 3, we will present an example which shows that Maskin's proof is incorrect when one of the hypotheses from Williams' result is not satisfied. In Section 4, we will prove that this additional hypothesis can be omitted from the list of sufficient conditions for Nash implementation.

2. Notation and Background Material

We assume that there are n agents (or players), where $n > 3$. For each $1 < j < n$, E_j will denote the set whose elements describe all possible characteristics of the j^{th} agent. The product $E = \prod_{i=1}^n E_i$ is the set of environments. We will use (e_i) to denote an arbitrary element of E .

A will denote the set of social alternatives. In order to avoid trivial cases, we assume that A has cardinality strictly greater than one.

For every $1 < j < n$, each element e_j of E_j is presumed to define a complete preordering [1, p. 8] on A . This represents the j^{th} agent's preferences over the elements of A when his characteristics are determined by e_j , so we will refer to it as a "preference preordering". Note that we do not make any assumption about the size of the set of preferences preorderings that is obtained as e_j varies over E_j . This aspect of the problem would depend upon the particular economic setting which is being modeled.

When agent j with characteristics e_j either prefers alternative a to alternative b or is indifferent between them, we will write " $a(>,e_j)b$," and when agent j strictly prefers a to b , we will write " $a(>,e_j)b$." $L(a,e_j)$ will represent the lower contour set of a in A when the preference preordering is determined by e_j :

$$L(a,e_j) = \{b \in A \mid a(>,e_j)b\}.$$

A top-ranked alternative for an agent with characteristics e_j is an element of A for which $L(a,e_j)$ equals A .

A performance standard F is a correspondence from E to A whose value at any environment is a nonempty subset of A . F is surjective if for any a in A , there exists an environment (e_i) such that a is in $F((e_i))$.

A game is a Cartesian product $S = \prod_{i=1}^n S_i$ and a mapping g (the game rule) from $\prod_{i=1}^n S_i$ to A . The product S is the strategy set of the game, and S_j is the j^{th} agent's strategy set. For simplicity, we will use (S, g) to denote the game with strategy set S and game form g .

A Nash equilibrium of (S, g) for the environment (e_i) is a point (s_i^*) of S which has the following property:

for each value of j ,

$$\{a \in A \mid a = g(s_j, s_{-j}^*), \text{ some } s_j \in S_j\} \subset L(g((s_i^*)), e_j).$$

Here, (s_j, s_{-j}^*) denotes the element of S obtained by starting with (s_i^*) and changing the j^{th} component from s_j^* to s_j .

Let $\tau(\cdot)$ denote the correspondence from E to S whose value at (e_i) is the set of all Nash equilibria for (e_i) . The game (S, g) implements the performance standard F in Nash equilibrium if the following equation is true for every value of (e_i) :

$$g \circ \tau((e_i)) = F((e_i)).$$

This corresponds to the definition of "full implementation" in [7].

We can now begin to discuss sufficient conditions for Nash implementation. A performance standard F is monotonic if it has the following property for any $(e_i), (e_i')$ in E and a in $F((e_i))$:

$$\begin{aligned} &\text{if } L(a, e_j) \subset L(a, e_j') \text{ for each value of } j, \\ &\text{then } a \in F((e_i')). \end{aligned} \tag{1}$$

Note that (1) clearly depends upon which set is designated as the set of social alternatives. F satisfies no veto power if the following is true for all a in A and (e_i) in E :

if $L(a, e_j) = A$ for all but at most one value of j , then $a \in F((e_i))$.

A discussion of each of these topics can be found in [4].

The theorem concerning sufficient conditions for Nash implementation which Maskin [3, Thm. 5] actually attempted to prove is stated below. It differs from his statement of the theorem because we have included the hypotheses which he either explicitly or implicitly adds in the course of his proof.

THEOREM 1 (Maskin). Let F be a surjective performance standard from $\prod_{i=1}^n E_i$ to A , where $n > 3$ and A is finite. If F satisfies both monotonicity and no veto power, then there exists a game (S, g) which implements F in Nash equilibrium.

Maskin proves this result by defining a game rule g on the strategy set which is given below:

$$S \equiv \prod_{i=1}^n S_i, \text{ where} \quad (2)$$

$$S_i \equiv \{((e_i), a) \in E \times A \mid a \in F((e_i))\}.$$

The game rule g which he defines is supposed to have the following three properties at $(s_1, s_2, \dots, s_n)$ in S :

$$\text{i) if } s_\ell = s_2 = \dots = s_n = ((e_i), a), \text{ then } g(s_1, s_2, \dots, s_n) = a; \quad (3)$$

$$\text{ii) if } s_\ell = ((e_i), a) \text{ for all } \ell \neq j, \text{ then } g(\cdot, s_{-j}) \text{ maps } S_j \text{ onto } L(a, e_j); \quad (4)$$

$$\text{iii) if } s_\ell \neq s_k \text{ and } j \neq \ell, k, \text{ then } g(\cdot, s_{-j}) \text{ maps } S_j \text{ onto } A. \quad (5)$$

Theorem 1 then follows by an application of Theorem 4 of [3], wherein it is proven that any game (S, g) which satisfies (2)-(5) will implement F in Nash equilibrium.

The application of the results of [7] to this problem leads to a slightly different result. Before we can state it, however, we must discuss an additional hypothesis. In order to prove a result which applies to sets A of arbitrary size, we make the following assumption:

there exists a mapping P_A from A^n to A such that
 for any (a_j^*) in A^n and for each value of j , the mapping (6)
 $P_A(\cdot, a_{-j}^*) : A \rightarrow A$ is surjective.

Here, $P_A(a, a_{-j}^*) \equiv P_A(a_1^*, \dots, a_{j-1}^*, a, a_{j+1}^*, \dots, a_n^*)$ at each a in A .

As mentioned above, this hypothesis is critical to any attempt to generalize Maskin's construction to sets A of arbitrary size. Hypothesis (6) will be satisfied whenever A can be given the structure of a group [5, p. 143] (e.g., when A is finite, countable, or even a Banach space). It can also be satisfied when A does not have an obvious group structure. It is an important hypothesis, since our assumption that F is surjective forces us to regard certain sets which have nontrivial geometric structure as the set of social alternatives. This can happen in the most common economic models (see, for instance, Section 11 of [7]). In these cases, it may be very difficult to find a suitable mapping P_A .

We can now state the result which follows from [7]. This theorem is an application of the construction process which is outlined in the proof of Theorem 4 of [7] to the direct revelation mechanism. The details of this application are extensively discussed in Section 9 of that paper.

In the statement of this theorem, we will use $F^{-1}(a)$ to denote the preimage of any alternative a :

$$F^{-1}(a) \equiv \{(e_i) \mid a \in F((e_i))\}.$$

THEOREM 2 (Williams). Let F be a surjective performance standard from $\prod_{i=1}^n E_i$ to A , where $n \geq 3$. If F satisfies the following properties, then there exists a game (S, g) which implements F in Nash equilibrium:

- i) F is monotonic;
- ii) F satisfies no veto power;
- iii) (6) holds;
- iv) if $n \geq 4$ and $a \in A$, then $F^{-1}(a)$ contains at least two distinct environments; if $n = 3$ and $a \in A$,

$$\text{then } F^{-1}(a) \text{ contains at least three distinct environments.} \quad (7)$$

Property (7) is needed to overcome a difficulty which can arise when the construction process from [7] is applied to F . Without this assumption, it is not clear that the game form which is constructed in the proof will satisfy property (5). This is discussed in Sections 8 and 9 of [7].

In Theorem 7 of that paper, it is shown that the following hypothesis can be substituted for (7) in the above theorem:

- for any a in A and any e_j in E_j , the set
- $$\{a \in A \mid b(>, e_j)a\} \text{ is either empty or contains at least three elements of } A. \quad (8)$$

This hypothesis is satisfied, for instance, when A is a topological manifold of dimension greater than or equal to one, and the preference preorderings on A are continuous [1, p. 56].

The construction process in [7] was clearly inspired by Maskin's: it follows the pattern that is given in (2)-(5). When A is finite, these two procedures are virtually identical. One would therefore expect that the difficulty which forced Williams [7] to assume either (7) or (8) in his construction would persist when Maskin's procedure is used. This in fact is true, and it will be demonstrated by the example in the following section.

3. A Counter-Example to Maskin's Construction

In this section we will apply Maskin's construction process to a performance standard which satisfies all of the hypotheses of Theorem 1, but which satisfies neither (7) nor (8). We will then show the game which is constructed by Maskin's method does not implement this performance standard in Nash equilibrium. The reader will note that the game form of this game does not satisfy (5).

We assume that there are three agents, and two alternatives -- $A = \{a_0, a_1\}$. For $j = 2, 3$, $E_j = \{>, <\}$, where $>$ denotes " a_0 is strictly preferred to a_1 ", while $<$ denotes " a_1 is strictly preferred to a_0 ." E_1 is assumed to contain a single point -- $E_1 = \{>\}$.

Our performance standard F is given by simple majority rule: for (e_1, e_2, e_3) in $\prod_{i=1}^3 E_i$,

$$F(e_1, e_2, e_3) = \begin{cases} a_1 & \text{if } e_2, e_3 = < \\ a_0 & \text{otherwise.} \end{cases}$$

It is easy to verify that F satisfies both monotonicity and no veto power, and F is clearly surjective. F does not satisfy (3), since $F^{-1}(a_1)$ is a singleton, and F does not satisfy (4), since A has cardinality two.

The game which Maskin constructs uses $\prod_{i=1}^3 S_i$ as its space of strategies, where the strategies that are available to the j^{th} agent are given below:

$$S_j = \{(e_1, e_2, e_3, a) \mid a \in F(e_1, e_2, e_3)\}.$$

His game form g is defined by the following formulas: for (s_1, s_2, s_3) in $\prod_{i=1}^3 S_i$,

$$i) \text{ if } s_1 = s_2 = s_3 = (e_1, e_2, e_3, a_t), \text{ then } g(s_1, s_2, s_3) = a_t; \quad (9)$$

ii) if $s_i = s_j = (e_1, e_2, e_3, a_t)$ but $s_i, s_j \neq s_k = (e_1^*, e_2^*, e_3^*, a_u)$,
then $g(s_1, s_2, s_3)$ is given below:

$$g(s_1, s_2, s_3) = \begin{cases} a_t & \text{if } L(a_t, e_k) = \{a_t\} ; \\ a_{t+u(\text{mod } 2)} & \text{if } L(a_t, e_k) = \{a_t, a_u\} . \end{cases} \quad (10)$$

iii) if $s_i = (e_1, e_2, e_3, a_t)$, $s_j = (\bar{e}_1, \bar{e}_2, \bar{e}_3, a_v)$, and $s_k = (e_1^*, e_2^*, e_3^*, a_u)$, where $s_i \neq s_j$, $s_j \neq s_k$, and $s_k \neq s_i$,
then $g(s_1, s_2, s_3) = a_{t+u+v(\text{mod } 2)} . \quad (11)$

For a review of addition modulo 2, consult McCoy [5, p. 96], or any elementary book on group theory.

Formulas (9), (10), and (11) are applications of the formulas (6), (7), and (8) [3, p. 22], respectively, of Maskin's construction to our example.¹ The reader should note that Maskin's formula (7) does not precisely determine our formula (10) as we have stated it above. His formula (7) depends upon the choice of an arbitrary numbering of the elements of the set $L(a_t, e_k)$ when $L(a_t, e_k) = \{a_t, a_u\}$.² Formula (10) is obtained when the most natural numbering is imposed upon $\{a_t, a_u\}$: a_0 is the zero element, and a_1 is the first element. If a_1 is counted as the zero element and a_0 as the first element, then Maskin's formula (7) would replace (10) with the following formula:

ii)' if $s_i = s_j = (e_1, e_2, e_3, a_t)$ but $s_i, s_j \neq s_k = (e_1^*, e_2^*, e_3^*, a_u)$,
then $g(s_1, s_2, s_3)$ is given below:

$$g(s_1, s_2, s_3) = \begin{cases} a_t & \text{if } L(a_t, e_k) = \{a_t\} , \\ a_{-(t+u)(\text{mod } 2)} & \text{if } L(a_t, e_k) = \{a_t, a_u\} . \end{cases} \quad (12)$$

¹ There is a misprint in some versions of Maskin's [3] paper. The formula for $g(\bar{s}_1, \dots, \bar{s}_n)$ in his (7) should read as follows: $g(\bar{s}_1, \dots, \bar{s}_n) = b([r + s] | m(a(r), \bar{R}_i); a, \bar{R}_i)$. We can see that this is what Maskin intended by noting the relationship of (7) to hypothesis (4) of his Theorem 4 [3, p. 19].

² See for the formula for $L(a, R_i)$ that appears at the bottom of p. 21 in [3].

Our analysis of this example depends critically upon the use of formula (10) rather than (12). We will return to this issue once we have completed this analysis.

Let $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ denote the following profile of strategies:

$$\begin{aligned}\bar{s}_1 &= \bar{s}_3 = (>, <, <, a_1) \\ \bar{s}_2 &= (>, >, >, a_0) .\end{aligned}$$

By formula (6), $g(\bar{s}_1, \bar{s}_2, \bar{s}_3) = a_1 + 0(\text{mod } 2) = a_1$. In a moment, however, we will show that $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is a Nash equilibrium for the profile of preferences $(e_1, e_2, e_3) = (>, <, >)$. Since the value of F at $(>, <, >)$ is a_0 , we will thereby show that the game form defined by (9), (10) and (11) does not implement F in Nash equilibrium.

In Table 1, we list the alternatives which the first agent can implement when he varies his strategy away from $\bar{s}_1$ while the second and third agents' strategies remain fixed at $\bar{s}_2$ and $\bar{s}_3$, respectively. Similarly, Table 2 lists the alternatives which the third agent can implement when he varies his strategy away from $\bar{s}_3$ while $\bar{s}_1$ and $\bar{s}_2$ remain fixed. In each of these tables, the "rule for computing g " refers to the formula ((9), (10), or (11)) of this paper which has been used to determine the value of g .

Table 1. The alternatives which Agent 1 can implement by varying his strategy away from $\bar{s}_1$ (when $s_2 = \bar{s}_2$, and $s_3 = \bar{s}_3$).

Agent 1	Agent 2	Agent 3	rule for computing g	value of g
s_1	$s_2 (= \bar{s}_2)$	$s_3 (= \bar{s}_3)$		
$>, <, <, a_1$	$>, >, >, a_0$	$>, <, <, a_1$	(10)	a_1
$>, >, >, a_0$	$>, >, >, a_0$	$>, <, <, a_1$	(10)	a_1
any other value for s_1	$>, >, >, a_0$	$>, <, <, a_1$	(11)	a_1

Table 2. The alternatives which Agent 3 can implement by varying his strategy away from $\bar{s}_3$ (when $s_1 = \bar{s}_1$, and $s_2 = \bar{s}_2$).

Agent 1	Agent 2	Agent 3	rule for computing g	value of g
$s_1 (= \bar{s}_1)$	$s_2 (= \bar{s}_2)$	s_3		
$>, <, <, a_1$	$>, >, >, a_0$	$>, <, <, a_1$	(10)	a_1
$>, <, <, a_1$	$>, >, >, a_0$	$>, >, >, a_0$	(10)	a_1
$>, <, <, a_1$	$>, >, >, a_0$	any other value for s_3	(11)	a_1

We will now verify that $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is a Nash equilibrium for the profile of preferences $(e_1, e_2, e_3) = (>, <, >)$. The value of g at $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is a_1 , so $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is clearly a Nash equilibrium for the second agent: he prefers a_1 to a_0 , so he could not implement a more preferred alternative by varying his strategy. Agent 1 prefers a_0 to a_1 , but by Table 1 we can see that he cannot implement a_0 by varying his own strategy away from $\bar{s}_1$. Therefore, $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is also a Nash equilibrium for the first agent. By using Table 2, a similar argument can be made to show that $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is a Nash equilibrium for the third agent when he prefers a_0 to a_1 . This completes our verification that $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is a Nash equilibrium for the profile $(>, <, >)$, and it also marks the end of the actual presentation of our example.

Our example might be criticized on several grounds, so we will now attempt to respond to some of this criticism. We first return to an issue that was raised earlier -- our use of (10) rather than (12) to define g . The reader should note that if (12) is used to define g , then $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ will no longer be a Nash equilibrium for the profile of preferences $(>, <, >)$. Problems might arise elsewhere in this example if (12) were used instead of (10), but it is conceivable

that redefining g in this manner might insure that the resulting game implements F in Nash equilibrium. It is also possible that a combination of (10) and (12) might be needed to properly define g .

If we have chosen to use a definition of g which causes Maskin's construction process to fail, then what is the point of this example? Our point is to illustrate a difficulty that arises in this construction, and that Maskin's proof (as given in [3]) fails to handle this difficulty. This issue is extensively discussed in Williams [7], so we will only note here that [7] handles the difficulty by adding either hypothesis (7) or (8) to Maskin's Theorem 5. In any particular example, it may be possible to define a game form on the same strategy space used in Maskin's construction in a way which will guarantee that the resulting game implements the performance standard in Nash equilibrium. If we are to use this construction to prove Theorem 1, however, we must show that it is always possible to define g in a suitable manner. Our example shows that Maskin's proof fails to do this.

This example might also be criticized because E_1 is a singleton -- $E_1 = \{>\}$. Maskin's result explicitly states that there are at least three agents, and the case when E_1 is a singleton seems perilously close to the two agent case. Also, even though Maskin does not rule out the case where E_1 is a singleton, it does make the example seem very contrived.

We will respond to this criticism in two ways. First, we acknowledge that the example is contrived, but it seems that only a contrived example would satisfy neither (7) nor (8). For instance, if the following standard hypotheses from voting theory are satisfied, then the performance standard F would satisfy (7):

- i) there are at least three agents ($n > 3$); (14)

ii) for every a in A and for each value of j , there exists an element e_j of E_j such that a is a top-ranked alternative relative to the preference ordering which is determined by e_j ; (15)

iii) F satisfies no veto power. (16)

This can be proven by applying no veto power to the different profiles of preferences for which at least $n - 1$ of the agents have a as a top-ranked alternative. Note that (15) is satisfied if each E_j determines all possible preference preorderings on A .

Second, we note that this example is merely intended to illustrate a flaw in Maskin's proof. The difficulty which Maskin overlooks is carefully described in full generality in [7]. A clear understanding of this difficulty indicates that it would persist in problems which involve more agents, more alternatives, and a richer class of environments, and that it does not hinge upon the particular features of our example. Because of the complexity of the larger problems, they are not very suitable for examples; for the sake of simplicity, we are therefore limited to elementary examples which seem contrived.

One more remark should be made about our example. The performance standard in this example can be implemented in Nash equilibrium. Extend the set E_1 to the larger set $\{>, <\}$, and extend the performance standard F -- "majority rule" -- to this new space of environments. As noted above, it is easy to verify that this extension of F will now satisfy (7) in addition to monotonicity and no veto power. Apply Maskin's construction to define a game which implements the extension of F . When we return to the restricted class of environments that is given in our example, this game will still implement F in Nash equilibrium.

4. Hypotheses (7) and (8) can be Omitted

Example 2 in [3] shows that neither (7) nor (8) is a necessary condition for Nash implementation. We now turn to the following problem: is Theorem 2 still true without assuming either (7) or (8)? The answer to this question is "yes", and we will prove it in this section. We will thereby prove a precise version of the result which Maskin initially stated in Theorem 5 of [3].

Our emphasis in this section is on (7) rather than (8), and our approach is motivated by the discussion at the very end of the last section. Let F be a performance standard on $E = \prod_{i=1}^n E_i$ which satisfies monotonicity and no veto power, but not (7). It must be true that for some alternative a in A and at least one value of the index j , agent j would never have a as a top-ranked alternative, no matter what value in E_j determines his characteristics. If this were false, i.e., if for any a in A and any $1 \leq j \leq n$, there exists a value e_j in E_j such that a is a top-ranked alternative relative to the preference ordering which is determined by e_j , then we could apply (14)-(16) to conclude that (7) must be satisfied by F .

This suggests the following approach. Beginning with a performance standard which satisfies monotonicity and no veto power but not (7), we will extend this performance standard to a larger set of environments in such a way that the extension will still satisfy monotonicity and no veto power, but it will also satisfy (7). The preceding discussion indicates that there are certain preference orderings which we should try to "add" to each E_j . The challenge, of course, will be to prove that a suitable extension of F to a larger class of environments actually exists. Once this extension has been obtained, however, we will then apply the construction which is outlined in [7] to this new performance standard. When the agents that are characterized by the points of the original set of environments are allowed to play the game which is constructed

by this procedure, the original performance standard will be implemented in Nash equilibrium. This is essentially what we did at the end of Section 3 to show that the performance standard in the example can be implemented in Nash equilibrium.

To carry out this argument, we must make an assumption about the original set of environments $E = \prod_{i=1}^n E_i$. We assume that there exists a set $\tilde{E} = \prod_{i=1}^n \tilde{E}_i$ which has the following properties:

- i) $E_j \subset \tilde{E}_j$ for each value of j ;
- ii) given any preference preordering on A and any value of j , there exists an element of $\tilde{E}_j$ which determines that preordering.

It is almost inconceivable that a set $\tilde{E}$ with these properties would not exist, but because the elements of each E_j may contain other information besides the preordering which they determine on A , it is not immediately clear how the existence of $\tilde{E}$ could be proven. For this reason, we are stating the existence of $\tilde{E}$ as a hypothesis. If for each value of j , every element e_j in E_j only determines a preordering on A and contains no other information, then the existence of $\tilde{E}$ is obvious. This is also true if the information that is given by any element e_j can be represented as an ordered pair whose first component determines a preordering on A , and whose second component contains the rest of the information that is contained in e_j . If information is encoded into the elements of each E_j in some more complicated fashion, then the existence of $\tilde{E}$ is unclear. In most applications, however, the existence of $\tilde{E}$ would not be an issue.

For the sake of precision, we also make the following definition. Given a performance standard F from E to A and a subset E' of $\tilde{E}$ which contains

E , an extension of F to E' is any performance standard from E' to A which equals F at every point of E .

Our argument will be divided into three steps. We begin by assuming that F is a performance standard from E to A which satisfies monotonicity and no veto power but not (7), and that E' is some arbitrary subset of $\tilde{E}$ which contains E . Theorem 3 contains necessary and sufficient conditions on F and E' for the existence of an extension of F to E' which satisfies monotonicity and no veto power. This result will be used in our proof of Theorem 4, wherein we will prove the existence of a subset E^* of $\tilde{E}$ and an extension F^* of F to E^* which satisfies monotonicity, no veto power, and (7). As the preceding discussion indicates, the actual proof of Maskin's result in Theorem 5 will be almost trivial once Theorem 4 has been proven.

Our proof of Theorem 3 will be simplified if we first make a few remarks. Let (e'_i) denote an arbitrary element of E' , the subset of $\tilde{E}$ over which we wish to extend F . Since we want the extension F' of F to be monotonic, we clearly could not assign the alternative a to (e'_i) if there exists an element (e_i) of E with the following property:

$$\begin{aligned} L(a, e'_i) &< L(a, e_j) \text{ for each value of } j, \\ \text{but } a &\notin F((e_i)). \end{aligned} \tag{17}$$

This tells us which alternatives must be excluded from $F'((e'_i))$.

We define the correspondence ρ from $\tilde{E}$ to A by the following formula:

for (e'_i) in $\tilde{E}$,

$$\rho((e'_i)) \equiv \{a \in A \mid \exists (e_i) \in E \text{ which satisfies (13)}\}.$$

In this definition, we are of course allowing for the possibility that the value of ρ at some element of $\tilde{E}$ might be the empty set. If F' is an extension

of F to E' which is monotonic, then clearly $F'((e'_i))$ must be a subset of $A \setminus \rho((e'_i))$ for all (e'_i) in $E' \setminus E$.

In the proof of Theorem 3, we will need to construct a monotonic extension F' of F to E' . In light of the preceding remarks, the following definition is a natural candidate:

$$\text{for } (e'_i) \text{ in } E', F'((e'_i)) \equiv \begin{cases} F((e'_i)) & \text{if } (e'_i) \in E; \\ A \setminus \rho((e'_i)) & \text{if } (e'_i) \in E' \setminus E. \end{cases} \quad (18)$$

If $\rho((e'_i))$ is a proper subset of A for every (e'_i) in $E' \setminus E$, then (18) defines an extension of F from E to E' . This motivates the following theorem.

THEOREM 3. Let F be a performance standard from E to A which satisfies both monotonicity and no veto power. Also, let E' be a subset of $\tilde{E}$ which contains E . There exists an extension F' of F to E' which satisfies monotonicity and no veto power if and only if $\rho((e'_i))$ is a proper subset of A for every (e'_i) in $E' \setminus E$.

PROOF. The proof of necessity follows immediately from the discussion which precedes the statement of this theorem. As mentioned above, any monotonic extension F' of F to E' must satisfy $F'((e'_i)) \subset A \setminus \rho((e'_i))$ at any $((e'_i))$ in $E' \setminus E$. If $\rho((e'_i))$ equals A for some (e'_i) in $E' \setminus E$, then F' would not be a well-defined performance standard.

We now prove sufficiency. An extension F' of F to E' is given by (18). We must verify that this F' satisfies both monotonicity and no veto power.

Let $(e'_i), (e''_i)$ be elements of E' , and let a be an element of $F'((e'_i))$ such that $L(a, e'_j)$ is a subset of $L(a, e''_j)$ for each value of j . To verify

that F' is monotonic, we must show that a is also an element of $F'((e_i''))$. We assume that it is not in this set, and we will obtain a contradiction.

Regardless of whether or not (e_i'') is in E , there exists an environment (e_i) in E which has the following property:

$$\begin{aligned} L(a, e_j'') &\subset L(a, e_j) \text{ for each value of } j, \\ \text{but } a &\notin F((e_i)). \end{aligned}$$

If (e_i'') is in E , one simply sets (e_i) equal to (e_i'') , and if (e_i'') is in $E' \setminus E$, the existence of (e_i) is guaranteed by the definition of F' . Since $L(a, e_j')$ is contained in $L(a, e_j'')$ for each value of j , the following statement must be true:

$$\begin{aligned} L(a, e_j') &\subset L(a, e_j) \text{ for each value of } j, \\ \text{but } a &\notin F((e_i)). \end{aligned}$$

If (e_i') is in E , then this statement contradicts the fact that F is monotonic, and if (e_i') is in $E' \setminus E$, then it contradicts the definition of F' . We have therefore shown that F' is monotonic.

We will not verify that F' satisfies no veto power. Let (e_i') be an element of E' and a an element of A such that for all but at most one value of the index j , $L(a, e_j')$ equals A . We assume that a is not in $F((e_i'))$, and we will obtain a contradiction.

Regardless of whether or not (e_i') is in E , there exists an element (e_i) in E which has the following property:

$$\begin{aligned} L(a, e_j') &\subset L(a, e_j) \text{ for each value of } j, \\ \text{but } a &\notin F((e_i)). \end{aligned} \tag{19}$$

As above, if (e_i') is in E , set (e_i) equal to (e_i') ; otherwise, the

existence of (e_i) is guaranteed by the definition of F' . By our hypothesis about a and (e_i') , it is clear that $L(a, e_j)$ must equal A for all but at most one value of the index j . Since (e_i) is in E and F satisfies no veto power, a must therefore be an element of $F((e_i))$. This contradicts (19), so we can conclude that F' satisfies no veto power.

Q.E.D.

We are now ready to state and prove the theorem which is the most critical to our analysis. In this theorem, we will prove that a performance standard F which satisfies monotonicity and no veto power can be extended in such a way that the extension still has both of these properties, but also satisfies (7).

The discussion at the beginning of this section indicates that if the performance standard F does not satisfy (7), then there are certain environments which we should try to add to E . This suggests a constructive procedure for proving the next theorem. If F does not satisfy (7), there exists an alternative a and a value of j such that a is not a top-ranked alternative for any of the preference preorderings that are determined by the elements of E_j . Choose an element e_j'' of $\tilde{E}_j$ such that a is a top-ranked alternative relative to the preference preordering that is determined by e_j'' , and define E'' to be the following subset of $\tilde{E}$:

$$\begin{aligned} E_i'' &\equiv E_i \quad \text{if } i \neq j; \\ E'' &= \prod_{i=1}^n E_i'', \text{ where} \\ E_j'' &= E_j \cup \{e_j''\}. \end{aligned}$$

In our proof of Theorem 4, we will apply Theorem 3 to show that there exists an extension of F to E'' which satisfies both monotonicity and no veto power. If F'' does not satisfy (7), then this procedure should be repeated using F'' and E'' instead of F and E . If A is finite, then in a finite number steps an extension of F will be obtained which satisfies monotonicity, no veto

power, and (7). Our proof of Theorem 4 is an abstraction of this construction, where Zorn's Lemma [2, p. 33] is applied to insure that this approach will be successful when the set of alternatives is of arbitrary size.

THEOREM 4. Let F be a performance standard from E to A . If F satisfies both monotonicity and no veto power, then there exists a subset $E^* = \prod_{i=1}^n E_i^*$ of $\tilde{E}$ which contains E and an extension F^* of F to E^* which satisfies monotonicity, no veto power, and (7).

Proof. Let $\underline{E}$ denote the set consisting of all subsets E' of $\tilde{E}$ which have the following properties:

$$i) \ E' = \prod_{i=1}^n E'_i, \text{ where } E_j \subset E'_j \text{ for each value of } j; \quad (20)$$

$$ii) \text{ there exists an extension } F' \text{ of } F \text{ to } E' \text{ which} \quad (21) \\ \text{satisfies monotonicity and no veto power.}$$

$\underline{E}$ is clearly nonempty, since E itself is an element of $\underline{E}$.

We will prove this theorem by the following argument. Zorn's Lemma will be applied to prove the existence of a maximal element of $\underline{E}$ -- an element E^* of $\underline{E}$ which is not a proper subset of any other element of $\underline{E}$. We will then prove that the extension F^* of F to E^* (whose existence is guaranteed by (21)) must satisfy (7).

We must first verify that $\underline{E}$ satisfies the hypothesis of Zorn's Lemma. Let $\underline{N}$ be any subset of $\underline{E}$ which is linearly ordered by the inclusion relation, i.e., if E', E'' are elements of $\underline{N}$, then either E' is a subset of E'' , or E'' is a subset of E' . We must show that $\underline{N}$ has an upper bound in $\underline{E}$.

Define N to be following subset of $\tilde{E}$:

$$N \equiv \bigcup_{E' \in \underline{N}} E'. \quad (22)$$

N is clearly an upper bound for $\underline{N}$, so Zorn's Lemma will be verified if we can show that N is an element of $\underline{E}$. N clearly has the structure of a Cartesian product: if we represent any element E' of $\underline{N}$ as $E' = \prod_{i=1}^n E'_i$, then since $\underline{N}$ is linearly ordered,

$$N = \prod_{i=1}^n \left(\bigcup_{E' \in \underline{N}} E'_i \right)$$

This shows that N satisfies (20).

We must also show that there exists an extension of F to N which satisfies both monotonicity and no veto power. By Theorem 3, we only need to prove that for any (e'_i) in $N \setminus E$, $\rho((e'_i))$ is a proper subset of A . By our definition of N in (22), any element (e'_i) of $N \setminus E$ must be in $E' \setminus E$ for some E' in $\underline{N}$. By (21), there exists an extension of F to E' which satisfies both monotonicity and no veto power. Theorem 3 then implies that $\rho((e'_i))$ must be a proper subset of A .

Zorn's Lemma can therefore be applied to $\underline{E}$. Let $E^* = \prod_{i=1}^n E_i^*$ be a maximal element of $\underline{E}$, and let F^* denote an extension of F to E^* which satisfies both monotonicity and no veto power. We will now show that F^* also satisfies (7).

Our proof will be by contradiction. If F^* does not satisfy (7), then there exists an element a in A and a value of j such that a is never a top-ranked alternative for the j^{th} agent, no matter what element in E_j^* determines his characteristics. We now follow the procedure which was outlined immediately before the statement of this theorem. Choose an element e_j^{**} of $\tilde{E}_j$ that determines a preference preordering on A which has a as a top-ranked alternative, and define the set E^{**} by the following formula:

$$E^{**} \equiv \prod_{i=1}^n E_i^{**}, \text{ where } \begin{aligned} E_i^{**} &\equiv E_i^* \text{ if } i \neq j; \\ E_j^{**} &= E_j^* \cup \{e_j^{**}\}. \end{aligned} \quad (23)$$

We will now prove that E^{**} is in $\underline{E}$ by showing that there exists an extension F^{**} of F to E^{**} which satisfies monotonicity and no veto power. This will contradict the maximality of E^* in $\underline{E}$, and we will thereby prove that F^* does indeed satisfy (7). This will complete the proof of this theorem.

By Theorem 3, we only need to show that $\rho((e_i^!))$ is a proper subset of A for every $(e_i^!)$ in $E^{**} \setminus E$. Since there exists an extension of F to E^* , we know that this is true for all $(e_i^!)$ in $E^* \setminus E$. We therefore assume that $(e_i^!)$ is in $E^{**} \setminus E^*$.

We can complete this argument by showing that the element a which was selected above cannot be in $\rho((e_i^!))$. If a is in $\rho((e_i^!))$, then there exists an element (e_j) of E such that the following is true:

$$\begin{aligned} L(a, e_j^!) &\subset L(a, e_j) \text{ for all values of } j, \\ \text{but } a &\notin F((e_j)). \end{aligned} \tag{24}$$

Since $(e_i^!)$ is in $E^{**} \setminus E^*$, by (23) we can see that $e_j^!$ must equal e_j^{**} . Because a is a top-ranked alternative for the preference preordering that is determined by e_j^{**} , by (24) a must also be a top-ranked alternative for the preference ordering that is determined by e_j . We selected a and j , however, because no element of $E_j^!$ determines a preference preordering which has a as a top-ranked alternative. This contradiction implies that a is not in $\rho((e_i^!))$. Q.E.D.

We are now prepared to state and prove the main result of this paper. Our statement of this theorem includes two hypotheses which are not explicitly stated in Maskin's original version of this result [3, Thm. 5]: we assume that the performance standard is surjective, and we also assume that there exists a mapping P_A from A^n to A which has the property that is described in (6). These hypotheses are an intrinsic part of any construction which generalizes the one which Maskin outlined in (2)-(5).

THEOREM 5. Let F be a surjective performance standard from $\prod_{i=1}^n E_i$ to A , where $n \geq 3$. If F satisfies monotonicity and no veto power, and if (6) is satisfied, then there exists a game (S, g) which implements F in Nash equilibrium.

PROOF. If F satisfies (7), then the existence of a game which fully implements F in Nash equilibrium is guaranteed by Theorem 2. If F does not satisfy (7), we apply Theorem 4 to obtain an extension F^* of F from E to some larger class of environments E^* , where F^* satisfies monotonicity, no veto power, and (7). Theorem 2 can now be applied to construct a game which implements F^* in Nash equilibrium. This game will also implement F when the set of environments is restricted from E^* to E . Q.E.D.

As a last remark, note that if the performance standard F does not satisfy (7), then the game which Theorem 5 produces to implement F will use an even larger strategy set than the one which Maskin originally proposed as an upper bound for Nash implementation. Both Maskin's and Williams' construction use the following subset of $E \times A$ as the strategy set:

for $1 \leq j \leq n$,

$$S_j \equiv \{((e_i), a) \in E \times A \mid a \in F((e_i))\}.$$

If the performance standard F does not satisfy (7), then we first extend F to a larger set of environments before we apply the construction process. Consequently, a larger strategy set than the one which Maskin first proposed will be used in this case.

This raises the following question: is it ever necessary to use a strategy set which is strictly larger than one which Maskin originally suggested as an a priori upper bound? To the author's knowledge, this question remains unanswered.

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