**ERGODIZATION RATES FOR LINEAR FLOW
ON THE TORUS**

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Abstract. We estimate the rate at which the image of a point fills the flat Euclidean n -torus when subjected to linear flow satisfying certain nonresonance or near-nonresonance conditions arising naturally in perturbation theory for ODE's. The estimates are explicit and display dependence on all relevant parameters, but are not sharp. These results were used to estimate the maximum time until near-collision with nuclei for highly energetic charged particles traveling through crystals in directions “far from channeling directions,” and are presented here with complete proofs.

Key Words: Small divisors, small denominators, ergodic theory, perturbation theory.

1. INTRODUCTION

Consider the flow on the flat Euclidean n -torus $\mathbf{T}^n$ arising from the ODE

$$\frac{d\theta}{dt} = \alpha, \tag{1.1}$$

where $\theta \in \mathbf{T}^n$, $t \in \mathbf{R}$, and α is a fixed “direction vector” of unit length (denoted $\alpha \in S^{n-1}$). For simplicity, we denote this linear flow by $\{\alpha_t\}$ (a precise description of the one-parameter translation maps $\alpha_t : \mathbf{T}^n \rightarrow \mathbf{T}^n$ is given in the next section). It is well known (see, e.g., Arnol'd and Avez 1967 or Arnol'd 1983) that the behavior of the phase curves of (1.1) is highly dependent on the resonance properties of α : if α is *resonant* (i.e., there exists a nonzero integer vector $k \in \mathbf{Z}^n$ such that the ordinary scalar product $k \cdot \alpha = 0$), then the phase curves of (1.1) are degenerate (they densely fill tori of dimension strictly less than n), while if α is *nonresonant*, then the phase curves of (1.1) are transitive (they fill all of $\mathbf{T}^n$ densely). In perturbation problems, where the unperturbed systems are typically like (1.1), in order to control small divisors the direction vector α is often required to be not only nonresonant but also “far from resonance” (or perhaps only far from “low-order” resonances) by means

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of a Diophantine condition. It is the aim of this paper to show that these conditions guarantee a rapid filling or “ergodization” of the torus.

The estimates presented here arose out of a problem in the physics of particle-solid interactions, in which Nekhoroshev-like perturbation procedures (cf. Nekhoroshev 1977) show that energetic particle trajectories are nearly rectilinear when particles are injected into crystals in such a way that the 3-dimensional direction vector α (relative to crystal coordinates) satisfies a truncated Diophantine condition of the form $|k \cdot \alpha| \geq C|k|^{-s}$ for $0 < |k| \leq N$ (here $k \in \mathbf{Z}^3$ and $|k| = |k_1| + |k_2| + |k_3|$). The rate at which such trajectories fill $\mathbf{T}^3$ (i.e., a crystal cell) may then be used to estimate the maximum time until close encounter with nuclei for these particles. The hope is that these estimates will in turn be useful in the physics of ion implantation, where enhanced particle-nuclei interaction is desired in order to achieve proper distributions of dopant ions in crystals (the interested reader may consult Dumas 1988, 1990, or Dumas and Ellison 1990 for details).

Here the estimates of ergodization time are presented as independent mathematical results. They may have applications to both Hamiltonian and nonhamiltonian perturbation theory for ODE’s (see, e.g., Lochak and Meunier 1988 for a discussion of the role of ergodic theory in perturbation theory); or as the “next best thing” to Denjoy-Koksma type estimates for rigid rotations on $\mathbf{T}^n$, $n \geq 2$ (cf. Yoccoz 1985). These applications will undoubtedly require information about the uniform distribution of nonresonant linear flows.

The remainder of this article is organized as follows. Section 2 introduces the basic language and notation used in the sequel, allowing the idea of ergodization time for linear flow to be introduced and the main results (Theorems 1 and 2) to be stated in Section 3. Section 4 is concerned with the proofs of these theorems; this requires technical estimates associated with properties of a certain class of test functions. Because these estimates are relatively standard and the main ideas of the proofs can be understood without them, they are relegated to an appendix following the proofs.

2. NOTATION AND TERMINOLOGY

We will work on the flat n -torus ($n \geq 2$) with the usual Euclidean metric. We

will view the torus as the fundamental cube in $\mathbf{R}^n$

$$\overline{\chi} = \underbrace{\left[-\frac{1}{2}, \frac{1}{2}\right] \times \cdots \times \left[-\frac{1}{2}, \frac{1}{2}\right]}_{n \text{ times}} \quad (2.1)$$

with opposite faces identified. The unique coordinates $\theta = (\theta_1, \dots, \theta_n)$ of a point in $\mathbf{T}^n$ will belong to the half-open cube

$$\chi = \underbrace{\left[-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left[-\frac{1}{2}, \frac{1}{2}\right)}_{n \text{ times}}. \quad (2.2)$$

Although strictly speaking the coordinates θ belong to $\mathbf{R}^n$, we will abuse the notation slightly and identify points in $\mathbf{T}^n$ with their coordinates by writing $\theta \in \mathbf{T}^n$. In these coordinates, the standard universal covering projection $p : \mathbf{R}^n \rightarrow \mathbf{T}^n$ takes the form

$$p(x) = \left(\{x_1 + 1/2\} - 1/2, \dots, \{x_n + 1/2\} - 1/2 \right), \quad (2.3)$$

where $\{x_i\}$ denotes the fractional part of x_i ($\{x_i\} = x_i - [x_i]$, where $[x_i]$ is the greatest integer less than or equal to x_i). The covering map is now used to define operations on $\mathbf{T}^n$: each operation (multiplication by scalars, addition, additive inversion, and translation by points in $\mathbf{R}^n$) is simply the projection of the usual operation with coordinates in $\mathbf{R}^n$; e.g., the coordinates of the sum of two points $\theta, \phi \in \mathbf{T}^n$ are $p(\theta + \phi)$.

The flat metric d may now be defined for any pair of points $\theta, \phi \in \mathbf{T}^n$ by

$$d(\theta, \phi) = \|\theta - \phi\|, \quad (2.4)$$

where the norm is the usual Euclidean norm on $\mathbf{R}^n$. The simple expression for the metric results from the “centered” coordinates on the torus, and we see that the distance between points on the torus is at most $\sqrt{n}/2$.

For $0 \leq R \leq 1/2$, the metric allows us to define the open ball of radius R centered on θ in $\mathbf{T}^n$ by $\mathcal{B}_R(\theta) = \{\phi \in \mathbf{T}^n \mid d(\theta, \phi) < R\}$. For our purposes it will not matter that the boundary of the closed ball $\overline{\mathcal{B}}_{1/2}(\theta)$ intersects itself in n points; we will still make use of it.

Finally, if $\alpha \in S^{n-1}$ is a direction vector, a *rectilinear orbit* (called a *straight winding* in Arnol’d 1967, 1983) of $\mathbf{T}^n$ with direction vector α will mean the following. For each $t \in \mathbf{R}$, we first consider the one-parameter family of translation maps $\alpha_t :$

$\mathbf{T}^n \rightarrow \mathbf{T}^n$ defined by $\alpha_t(\theta) = \theta + t\alpha$. A rectilinear orbit of $\mathbf{T}^n$ with direction vector α and initial condition θ is then the image of θ under the linear flow defined by α_t over some closed interval $[a, b]$ in $\mathbf{R}$:

$$\bigcup_{a \leq t \leq b} \alpha_t(\theta). \quad (2.5)$$

3. MAIN RESULTS

3.1 The Idea of Ergodization Time for Rectilinear Flow

For nonresonant α , the denseness of $\bigcup_{0 \leq t < \infty} \alpha_t(\theta)$ implies an “ergodization” result; i.e., for nonresonant α , given $R > 0$, there exists a $T > 0$ such that no point in $\mathbf{T}^n$ is more than R -distant from any orbit $\bigcup_{0 \leq t \leq T} \alpha_t(\theta)$. It is easy to see that this condition is equivalent to the following

Definition. Given $R > 0$, the direction vector $\alpha \in S^{n-1}$ is said to ergodize $\mathbf{T}^n$ to within R after time T if

$$\bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\theta)) = \mathbf{T}^n \quad (3.1)$$

for all $\theta \in \mathbf{T}^n$.

In words, this is expressed by saying that given $R > 0$, the image of any ball of radius R under the flow generated by α between 0 and T is all of $\mathbf{T}^n$. The property of ergodizing the torus to within R after time T is independent of initial condition in the following sense: for any pair $\theta, \phi \in \mathbf{T}^n$, $\bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\theta)) = \bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\phi)) + \{\theta - \phi\}$, so that if the ergodization condition $\bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\theta)) = \mathbf{T}^n$ is satisfied for one $\theta \in \mathbf{T}^n$, then it is satisfied for all $\theta \in \mathbf{T}^n$.

If all that can be said about the direction vector α is that it is nonresonant, then for fixed R , the time T required to ergodize $\mathbf{T}^n$ is of course highly dependent on α , and may in general be arbitrarily long (think of a sequence of nonresonant α 's converging to a low-order resonant α). Furthermore, it is important to note that with the above definition of ergodization, even orbits given by resonant α can ergodize $\mathbf{T}^n$ for fixed $R > 0$, provided they are resonant at sufficiently high order (i.e., if $\alpha \cdot k = 0$ only for integer vectors k with large norm). This latter case is dealt with in Theorem 2 of Section 3.3 below.

3.2 The Case of Ordinary Diophantine Conditions

We first state the theorem on ergodization time for the case of highly nonresonant direction vectors; i.e., direction vectors satisfying ordinary Diophantine conditions. Specifically, for $s > n/2$ and $C > 0$, we consider α belonging to the set

$$\mathcal{D}(s, C) = \left\{ \alpha \in S^{n-1} \mid |\alpha \cdot k| > \frac{C}{\|k\|^s} \forall k \in \mathbb{Z}^n \setminus \{0\} \right\}, \quad (3.2)$$

where $\| \cdot \|$ designates the Euclidean norm. Direction vectors in $\mathcal{D}(s, C)$ are said to satisfy a Diophantine condition of order s . For $s > n/2$, it is not hard to show that $\mathcal{D}(s, C)$ is nonempty for small enough $C > 0$; in fact, the measure of the complement of $\mathcal{D}(s, C)$ (in S^{n-1}) vanishes with C . We then have:

Theorem 1. *Let $0 < R \leq 1$. Given any highly nonresonant direction vector $\alpha \in \mathcal{D}(s, C)$, rectilinear orbits of $\mathbf{T}^n$ with direction vector α will ergodize $\mathbf{T}^n$ to within R after time T , where*

$$T = \frac{2\|V_*\|_\Delta}{C\pi R^{s+n/2}} \quad (3.3)$$

is independent of α .

The constant $\|V_*\|_\Delta$ appearing above is a Sobolev norm of a certain “smoothest test function” with support in the unit (diameter) ball in $\mathbf{T}^n$; it depends only on n and s and its existence and uniqueness are discussed in the Appendix. Both the norm $\| \cdot \|_\Delta$ and the set of test functions (of which V_* is the unique element minimizing $\| \cdot \|_\Delta$) arise naturally in the course of the proof of Theorem 1 below.

3.3 The Case of Truncated Diophantine Conditions

It is intuitively clear that, for fixed $R > 0$, rectilinear orbits arising from resonant direction vectors will ergodize the torus, provided they are resonant at sufficiently high order. Such orbits are indeed degenerate, but at high enough order, they build a “web” fine enough to trap the n -ball of radius R . In fact, given R and given a Diophantine condition, we can determine a critical cutoff N_{crit} such that, when direction vectors resonant at orders beyond the cutoff are admitted, the ergodization time found in the untruncated case is retained up to a factor depending on the cutoff. Our method of proof yields the estimate

$$N_{\text{crit}} \leq N_0 \equiv \kappa_o R^{-\left(\frac{s+n/2}{s-n/2}\right)} + \sqrt{n}/2, \quad (3.4)$$

where the constant κ_o depends only on n and s and is given in Eq. (4.29).

We then enlarge the previous class of direction vectors $\mathcal{D}(s, C)$ by admitting vectors which are resonant at orders higher than N . We thus define $\mathcal{D}(s, C, N)$, the class of α 's satisfying a truncated Diophantine condition of order $s > n/2$, by

$$\mathcal{D}(s, C, N) = \left\{ \alpha \in S^{n-1} \mid |\alpha \cdot k| > \frac{C}{\|k\|^s} \quad \forall k \in \mathbf{Z}^n \text{ with } 0 < \|k\| \leq N \right\}. \quad (3.5)$$

We then have

Theorem 2. *Let $0 < R \leq 1$, and choose $N > N_0$.*

Then given $\alpha \in \mathcal{D}(s, C, N)$, $\mathbf{T}^n$ will be ergodized to within R by rectilinear orbits with direction vector α after time

$$T = \frac{2\|V_*\|_\Delta}{C\pi R^{s+n/2}} \left[1 - \left(\frac{N_0 - \sqrt{n}/2}{N - \sqrt{n}/2} \right)^{s-n/2} \right]^{-1/2}. \quad (3.6)$$

Remark 3.1. It should be pointed out that when the taxicab norm $\|k\| \equiv \sum_i |k_i|$ is used in place of the Euclidean norm in defining the Diophantine conditions (3.2) and (3.5), the following slight changes should be made in Theorems 1 and 2. The ergodization time (3.3) of Theorem 1 is multiplied by $n^{s/2}$ and so becomes

$$T = \frac{2n^{s/2}\|V_*\|_\Delta}{C\pi R^{s+n/2}}, \quad (3.7)$$

while the estimate N_0 of the critical cutoff N_{crit} is multiplied by $\sqrt{n}$ so that (3.4) becomes

$$N_{\text{crit}} \leq N^* \equiv \kappa R^{-\left(\frac{s+n/2}{s-n/2}\right)} + n/2, \quad \kappa \equiv \sqrt{n} \kappa_o, \quad (3.8)$$

and N^* takes the place of N_0 in Theorem 2. Finally, the ergodization time (3.6) for the truncated case of Theorem 2 should be replaced by

$$T = \frac{2n^{s/2}\|V_*\|_\Delta}{C\pi R^{s+n/2}} \left[1 - \left(\frac{N^* - n/2}{N - n/2} \right)^{s-n/2} \right]^{-1/2}. \quad (3.9)$$

The places where these changes arise will be pointed out parenthetically in the proofs.

Remark 3.2. The fact that the estimate (3.3) of ergodization time is not sharp may be seen by specializing to the two-dimensional case. For $n = 2$, elementary techniques (i.e., reduction of the flow on $\mathbf{T}^2$ to a rotation on $\mathbf{T}^1$; expansion of the rotation

number in continued fractions) provide estimates of the ergodization time of order $(CR^s)^{-1}$, which is better than the estimate of order $(CR^{s+1})^{-1}$ provided by Theorem 1. Unfortunately, because of the difficulty associated with adequate “simultaneous rational approximations” to the components of a rotation vector for rotations in $\mathbf{T}^n$, $n > 1$, techniques analogous to the continued fraction expansion apparently do not apply to this estimate for $n > 2$.

4. PROOFS OF THEOREMS 1 AND 2

4.1 Preliminaries for the Proof of Theorem 1

In preparation for the proof of Theorem 1, we first establish a simple lemma and corollary which reformulate the problem slightly in a way that will allow us to take advantage of the properties of the test functions discussed below. Again, $\mathcal{B}_R(\theta)$ will denote the n -ball of radius $R > 0$ in $\mathbf{T}^n$ centered on θ .

Lemma 4.1. *Let $\theta, \phi \in \mathbf{T}^n$, and let $T > 0$. Suppose there exist $s, t \in [0, T/2]$ such that $\alpha_t(\mathcal{B}_{R/2}(\theta)) \cap \alpha_{-s}(\mathcal{B}_{R/2}(\phi)) \neq \emptyset$. Then $\phi \in \alpha_{s+t}(\mathcal{B}_R(\theta))$.*

PROOF. Using the group and distance-preserving properties of $\{\alpha_t\}$, we have $\emptyset \neq C = \alpha_t(\mathcal{B}_{R/2}(\theta)) \cap \alpha_{-s}(\mathcal{B}_{R/2}(\phi))$, which implies that $\emptyset \neq \alpha_s(C) \subset \alpha_s(\alpha_t(\mathcal{B}_{R/2}(\theta))) \cap \alpha_s(\alpha_{-s}(\mathcal{B}_{R/2}(\phi))) = \alpha_{s+t}(\mathcal{B}_{R/2}(\theta)) \cap \mathcal{B}_{R/2}(\phi)$. Let z belong to this intersection. Then $|\alpha_{s+t}(\theta) - \phi| \leq |\alpha_{s+t}(\theta) - z| + |z - \phi| \leq R/2 + R/2 = R$, and so $\phi \in \alpha_{s+t}(\mathcal{B}_R(\theta))$ as desired. $\square$

An immediate consequence of Lemma 4.1 is the following

Corollary 4.1. *Suppose the hypotheses of Lemma 1 are satisfied for every pair of points $\theta, \phi \in \mathbf{T}^n$. Then $\bigcup_{0 \leq t \leq T} \alpha_t(\mathcal{B}_R(\theta)) = \mathbf{T}^n$; in other words, α ergodizes $\mathbf{T}^n$ to within R after time T .*

Now the idea for the proof of Theorem 1 is to trace the time averages of smooth functions with compact support as they evolve in opposite directions under the flow defined by an arbitrary direction vector in $\mathcal{D}(s, C)$. If the supports of these functions are concentrated inside balls of radius $R/2$ centered on two arbitrary points in $\mathbf{T}^n$, the supports of their evolving time averages do not necessarily intersect until the flow has ergodized $\mathbf{T}^n$ to within R . As long as the supports of the time averages are

disjoint, their inner product is zero, and this leads to a computable expression for the ergodization time. The small divisors in this expression are controlled by the Diophantine conditions on α , and this control gives an estimate of the ergodization time for all Diophantine α 's.

The test functions and their properties used in the proof are discussed in detail in the Appendix. To understand the idea of the proof, it is enough to note that the set of test functions is denoted $\mathcal{A}_{R/2}^s$ ($0 < R \leq 1$) and consists of functions $u : \mathbf{T}^n \rightarrow \mathbf{R}$ with the following properties:

- (i) u is continuous and nonnegative on its support contained in $\overline{\mathcal{B}}_{R/2}(0)$,
- (ii) $\int_{\mathbf{T}^n} u(x) dx = 1$, and
- (iii) $D^\beta u \in L^2(\mathbf{T}^n)$ for every multiindex β with $|\beta| \leq s$.

Given $u \in \mathcal{A}_{R/2}^s$ and $z \in \mathbf{T}^n$, $u_z(x) = u(x - z)$ will denote the translation of u by z ; the support of u_z has simply been shifted from the ball $\overline{\mathcal{B}}_{R/2}(0)$ centered at 0 to the ball $\overline{\mathcal{B}}_{R/2}(z)$ centered at z . The set of all translates of functions in $\mathcal{A}_{R/2}^s$ will be denoted by $\mathcal{TA}_{R/2}^s$.

For $u_z \in \mathcal{TA}_{R/2}^s$, we define $\langle u_z \rangle_t^\alpha : \mathbf{T}^n \rightarrow \mathbf{R}$, the t -average of u_z in the direction α , by

$$\langle u_z \rangle_t^\alpha(x) = \frac{1}{t} \int_0^t (u_z \circ \alpha_r)(x) dr. \quad (4.1)$$

It is important here to see that

$$\text{supp}(\langle u_z \rangle_t^\alpha) \subset \bigcup_{0 \leq r \leq t} \alpha_{-r}(\overline{\mathcal{B}}_{R/2}(z)), \quad (4.2)$$

which the reader is invited to check. If $u_y \in \mathcal{TA}_{R/2}^s$, and u is expressed in terms of its Fourier series as

$$u(x) = \sum_{k \in \mathbf{Z}^n} u_k e^{2\pi i k \cdot x} \quad (4.3)$$

then u_y may be written

$$u_y(x) = \sum_{k \in \mathbf{Z}^n} u_k e^{2\pi i k \cdot (x - y)}. \quad (4.4)$$

We thus compute

$$\langle u_y \rangle_t^\alpha(x) = \frac{1}{t} \int_0^t (u_y \circ \alpha_r)(x) = \frac{1}{t} \int_0^t \sum_{k \in \mathbf{Z}^n} u_k e^{2\pi i k \cdot (x - y - r\alpha)} dr$$

$$= \sum_{k \cdot \alpha = 0} u_k e^{2\pi i k \cdot (x-y)} + \frac{1}{t} \sum_{k \cdot \alpha \neq 0} u_k e^{2\pi i k \cdot (x-y)} \left(\frac{e^{-2\pi i t k \cdot \alpha} - 1}{-2\pi i k \cdot \alpha} \right). \quad (4.5)$$

4.2 Proof of Theorem 1

The proof now begins in earnest. Let $y, z \in \mathbf{T}^n$, $0 < R \leq 1$, $\alpha \in \mathcal{D}(s, C)$ and $u \in \mathcal{A}_{R/2}^s$ be arbitrary, and let $T > 0$ be such that the images of the balls $\bar{\mathcal{B}}_{R/2}(y)$ and $\bar{\mathcal{B}}_{R/2}(z)$ do not intersect under forward and reverse orbits in the direction of α between 0 and $T/2$:

$$\left(\bigcup_{0 \leq t \leq T/2} \alpha_{-t}(\bar{\mathcal{B}}_{R/2}(y)) \right) \cap \left(\bigcup_{0 \leq s \leq T/2} \alpha_s(\bar{\mathcal{B}}_{R/2}(z)) \right) = \emptyset. \quad (4.6)$$

Since the supports of $\langle u_y \rangle_t^\alpha$ and $\langle u_z \rangle_{-t}^\alpha$ are contained in

$$\bigcup_{0 \leq t \leq T/2} \alpha_{-t}(\bar{\mathcal{B}}_{R/2}(y)) \quad \text{and} \quad \bigcup_{0 \leq t \leq T/2} \alpha_t(\bar{\mathcal{B}}_{R/2}(z)), \quad (4.7)$$

respectively, the inner product

$$\left(\langle u_y \rangle_t^\alpha, \langle u_z \rangle_{-t}^\alpha \right) = \int_{\mathbf{T}^n} \langle u_y \rangle_t^\alpha(x) \overline{\langle u_z \rangle_{-t}^\alpha(x)} dx \quad (4.8)$$

vanishes for $0 \leq t \leq T/2$. Using the series expression (4.5) obtained above for the time averages of u_y and u_z , we directly compute the integral for $0 \leq t \leq T/2$:

$$\begin{aligned} 0 &= \left(\langle u_y \rangle_t^\alpha, \langle u_z \rangle_{-t}^\alpha \right) = \int_{\mathbf{T}^n} \sum_{\substack{k \cdot \alpha = 0 \\ l \cdot \alpha = 0}} u_k \bar{u}_l e^{2\pi i [(k-l) \cdot x - (k \cdot y - l \cdot z)]} dx \\ &\quad + \frac{1}{t} \int_{\mathbf{T}^n} \left[\sum_{\substack{k \cdot \alpha = 0 \\ l \cdot \alpha \neq 0}} u_k \bar{u}_l e^{2\pi i [(k-l) \cdot x - (k \cdot y - l \cdot z)]} \left(\frac{e^{2\pi i t l \cdot \alpha} - 1}{2\pi i l \cdot \alpha} \right) \right. \\ &\quad \left. + \sum_{\substack{k \cdot \alpha \neq 0 \\ l \cdot \alpha = 0}} u_k \bar{u}_l e^{2\pi i [(k-l) \cdot x - (k \cdot y - l \cdot z)]} \left(\frac{e^{2\pi i t k \cdot \alpha} - 1}{2\pi i k \cdot \alpha} \right) \right] dx \\ &\quad + \frac{1}{t^2} \int_{\mathbf{T}^n} \sum_{\substack{k \cdot \alpha \neq 0 \\ l \cdot \alpha \neq 0}} u_k \bar{u}_l e^{2\pi i [(k-l) \cdot x - (k \cdot y - l \cdot z)]} \left(\frac{e^{2\pi i t k \cdot \alpha} - 1}{2\pi i k \cdot \alpha} \right) \left(\frac{e^{2\pi i t l \cdot \alpha} - 1}{2\pi i l \cdot \alpha} \right) dx. \end{aligned} \quad (4.9)$$

Examination of the above integrals reveals that in each term, nonzero contributions can occur only when $k = l$. Since $k = l$ never occurs in the middle terms, they vanish, and after integration the sum of the first and last terms reduces to

$$\sum_{k \cdot \alpha = 0} |u_k|^2 e^{2\pi i k \cdot (y-z)} - \frac{1}{t^2} \sum_{k \cdot \alpha \neq 0} |u_k|^2 e^{2\pi i k \cdot (y-z)} \frac{(e^{2\pi i t k \cdot \alpha} - 1)^2}{4\pi^2 (k \cdot \alpha)^2}. \quad (4.10)$$

Because $\alpha \in \mathcal{D}(s, C)$, the only k for which $k \cdot \alpha = 0$ is $k = 0$. The first sum therefore reduces to $|u_0|^2 = \left| \int_{\mathbf{T}^n} u \right|^2 = 1$, and after multiplying by t^2 , we have

$$t^2 = \sum_{k \neq 0} |u_k|^2 e^{2\pi i k \cdot (y-z)} \frac{(e^{2\pi i t k \cdot \alpha} - 1)^2}{4\pi^2 (k \cdot \alpha)^2} \leq \frac{1}{\pi^2} \sum_{k \in \mathbf{Z}^n} \frac{|u_k|^2}{(k \cdot \alpha)^2}. \quad (4.11)$$

The Diophantine condition $|\alpha \cdot k| > C\|k\|^{-s}$ then applies to the small divisors to ensure that this sum is

$$< \frac{1}{C^2 \pi^2} \sum_{k \in \mathbf{Z}^n} \|k\|^{2s} |u_k|^2. \quad (4.12)$$

(Note: it is here that the factor $n^{s/2}$ enters in (3.7) if the taxicab norm is used in (3.2); this arises through the use of $|k| \leq \sqrt{n}\|k\|$ in arriving to (4.12).) Now $u \in \mathcal{A}_{R/2}^s$ was arbitrary, which implies that

$$t^2 < \frac{1}{C^2 \pi^2} \inf_{u \in \mathcal{A}_{R/2}^s} \sum_{k \in \mathbf{Z}^n} \|k\|^{2s} |u_k|^2. \quad (4.13)$$

As discussed in Subsection A.2 of the Appendix, the sum over which the infimum is taken in (4.13) is in fact the square of a Sobolev norm on the appropriate Sobolev space $H_0^s(B_{R/2}(0))$ containing $\mathcal{A}_{R/2}^s$; this norm is denoted $\|\cdot\|_\Delta$. Eq. (4.13) may thus be written

$$t^2 < \frac{1}{C^2 \pi^2} \inf_{u \in \mathcal{A}_{R/2}^s} \|u\|_\Delta^2. \quad (4.14)$$

In A.3, it is shown that the infimum is actually attained by the unique element $U \in \mathcal{A}_{R/2}^s$ whose “finite lift” U_* (cf. §A.1) has the same norm. Finally, part (i) of Lemma A.1 shows that the norm obeys the scaling relation $\|U_*\|_\Delta = R^{-s-n/2} \|V_*\|_\Delta$, where $\|V_*\|_\Delta$ is independent of R and is the unique element of $\mathcal{A}_{1/2}^{s*}$ (the set of finite lifts of test functions with support in the ball of unit diameter) which minimizes $\|\cdot\|_\Delta$. We thus have

$$t^2 < \frac{\|U_*\|_\Delta^2}{C^2 \pi^2} = \frac{\|V_*\|_\Delta^2}{C^2 \pi^2 R^{2s+n}}. \quad (4.15)$$

The proof is nearly complete. We have shown that if T is such that the nonintersection condition (4.6) holds, then for all $0 < t \leq T/2$, the inequality (4.15) is true. In particular, for $t = T/2$, it follows that $T^2 < 4\|V_*\|_\Delta^2 / (C^2 \pi^2 R^{2s+n})$, and so if $T = 2\|V_*\|_\Delta / (C\pi R^{s+n/2})$, then condition (4.6) must be false. In other words, it must be true that

$$\left(\bigcup_{0 \leq t \leq T/2} \alpha_{-t}(\overline{B}_{R/2}(y)) \right) \cap \left(\bigcup_{0 \leq s \leq T/2} \alpha_s(\overline{B}_{R/2}(z)) \right) \neq \emptyset, \quad (4.16)$$

in which case there exist $s, t \in [0, T/2]$ such that $\alpha_{-t}(\overline{\mathcal{B}}_{R/2}(y)) \cap \alpha_s(\overline{\mathcal{B}}_{R/2}(z)) \neq \emptyset$. Since y and z were arbitrary, Corollary 4.1 shows that

$$\bigcup_{0 \leq t \leq T} \alpha_t(\overline{\mathcal{B}}_{R/2}(z)) = \mathbf{T}^n, \quad (4.17)$$

and it follows that any $\alpha \in \mathcal{D}(s, C)$ ergodizes the torus to within R after time

$$T = \frac{2\|V_*\|_{\Delta}}{C\pi R^{s+n/2}} \cdot \square \quad (4.18)$$

4.3 Preliminaries for the Proof of Theorem 2

Once again, a few definitions and auxiliary results are necessary before stating and proving Theorem 2. First, given $N > 0$ and a multi-unit-periodic function u , we define $u^{\leq}$, the function u cut off at N , to be the function obtained from u by truncating its Fourier series at $\|k\| = N$:

$$u^{\leq}(x) = \sum_{\substack{k \in \mathbb{Z}^n \\ \|k\| \leq N}} u_k e^{2\pi i k \cdot x}. \quad (4.19)$$

We now state a very simple proposition on the approximate orthogonality of vectors in an inner product space, then use it to prove another lemma needed in the proof of Theorem 2.

Proposition 4.1. *Let α, β, a , and b be vectors in an inner product space which satisfy $(\alpha, \beta) = 0$, $\|\alpha\| = \|\beta\|$, $\|a\|, \|b\| \leq \|\alpha\|$, and $\|\alpha - a\|, \|\beta - b\| < Q$. Then $|(a, b)| < 2Q\|\alpha\|$.*

PROOF. The proof is immediate: $|(a, b)| = |(a, b) - (\alpha, b) + (\alpha, b) - (\alpha, \beta)| \leq |(a - \alpha, b)| + |(\alpha, b - \beta)| \leq \|a - \alpha\| \cdot \|b\| + \|\alpha\| \cdot \|b - \beta\| < 2Q\|\alpha\|$. $\square$

Lemma 4.2. *Let $u \in \mathcal{A}_{R/2}^s$, $\alpha \in S^{n-1}$, and suppose*

$$\|u - u^{\leq}\|_{L^2}^2 \leq \frac{\delta^2}{4\|u\|_{L^2}^2}.$$

Then $(\langle u_y \rangle_t^\alpha, \langle u_z \rangle_{-t}^\alpha) = 0 \implies |(\langle u_y^{\leq} \rangle_t^\alpha, \langle u_z^{\leq} \rangle_{-t}^\alpha)| \leq \delta$, where the inner product is the one defined in Eq. (4.8).

PROOF. We have

$$|(\langle u_y^{\leq} \rangle_t^\alpha, \langle u_z^{\leq} \rangle_{-t}^\alpha)| = \left| \int_{\mathbf{T}^n} \left(\frac{1}{t} \int_0^t u^{\leq}(x - y - r\alpha) dr \right) \overline{\left(\frac{1}{t} \int_0^t u^{\leq}(x - z + s\alpha) ds \right)} dx \right|$$

$$\begin{aligned}
&= \frac{1}{t^2} \left| \int_{\mathbf{T}^n} \left(\int_0^t \int_0^t u^\leq(x-y-r\alpha) \overline{u^\leq(x-z+s\alpha)} dr ds \right) dx \right| \\
&= \frac{1}{t^2} \left| \int_0^t \int_0^t (u_y^\leq \circ \alpha_r, u_z^\leq \circ \alpha_{-s}) dr ds \right|, \tag{4.20}
\end{aligned}$$

where the last equality uses Fubini's theorem. However, since the norm is translation invariant, we see that $\|u - u^\leq\|_{L^2}^2 \leq \delta^2/(4\|u\|_{L^2}^2) \implies \|u_y \circ \alpha_r - u_y^\leq \circ \alpha_r\|_{L^2}^2 \leq \delta^2/(4\|u\|_{L^2}^2)$ and $\|u_z \circ \alpha_{-s} - u_z^\leq \circ \alpha_{-s}\|_{L^2}^2 \leq \delta^2/(4\|u\|_{L^2}^2)$. Proposition 4.1 then applies to give $|(u_y^\leq \circ \alpha_r, u_z^\leq \circ \alpha_{-s})| \leq \delta$. Therefore the integral at the end of (4.20) is bounded by $\delta t^{-2} \int_0^t \int_0^t dr ds = \delta$, as desired. $\square$

4.4 Proof of Theorem 2

Theorem 2 will be proved by first stating and proving a lemma whose proof closely parallels that of Theorem 1. Theorem 2 will then follow immediately from this lemma.

We consider n and s fixed in advance; this determines the value of the constant K which is given explicitly in Lemma A.2 of the Appendix. We may then state

Lemma 4.3. *Let K be determined as in part (ii), Lemma A.2. Choose $0 < R \leq 1$, choose $0 < \delta < R^n/K$, and set*

$$N = M + \frac{\sqrt{n}}{2} = R^{-1} \left(\frac{4\|V_*\|_{L^2}^2}{\delta^2} \right)^{\frac{1}{2s-n}} + \frac{\sqrt{n}}{2}. \tag{4.21}$$

Then given $\alpha \in \mathcal{D}(s, C, N)$, $\mathbf{T}^n$ will be ergodized to within R by rectilinear orbits with direction vector α after time

$$T = \frac{2\|V_*\|_\Delta}{C\pi R^{s+n/2} \sqrt{1 - \delta K R^{-n}}}. \tag{4.22}$$

PROOF. The proof relies on the computation from the proof of Theorem 1. Assume the hypotheses of the lemma, and let $y, z \in \mathbf{T}^n$, $U \in \mathcal{A}_{R/2}^s$ be the minimizing test function of Eq. (4.15), and let $T > 0$ be as in the proof of Theorem 1 (i.e., such that the images of the balls $\overline{\mathcal{B}}_{R/2}(y)$ and $\overline{\mathcal{B}}_{R/2}(z)$ do not intersect under forward and reverse orbits in the direction of α between 0 and $T/2$). Now the L^2 -difference of U and $U^\leq$ (U cut off at $N = M + \sqrt{n}/2$) is estimated in part (ii) of Lemma A.2 as

$$\|U - U^\leq\|_{L^2}^2 = \sum_{\|k\| > N} |U_k|^2 \leq \frac{K^2}{R^{2s} M^{2s-n}} \tag{4.23}$$

which, using (4.21) and then Eq. (A.8) and part (ii) of Lemma A.1 is

$$= \frac{K^2 \delta^2}{4R^n \|V_*\|_{L^2}^2} = \frac{K^2 \delta^2}{4R^{2n} \|U\|_{L^2}^2}. \quad (4.24)$$

Since $(\langle U_y \rangle_t^\alpha, \langle U_z \rangle_{-t}^\alpha) = 0$ for $0 \leq t \leq T$, Lemma 4.2 shows that $|\langle \langle U_y^\leq \rangle_t^\alpha, \langle U_z^\leq \rangle_{-t}^\alpha \rangle| \leq \delta K R^{-n} < 1$ on the same time interval. Therefore, from the computations in Eqs. (4.9) and (4.10), we have

$$\frac{\delta K}{R^n} \geq \sum_{\substack{k \cdot \alpha = 0 \\ \|k\| \leq N}} |U_k|^2 e^{2\pi i k \cdot (y-z)} - \frac{1}{t^2} \sum_{\substack{k \cdot \alpha \neq 0 \\ \|k\| \leq N}} |U_k|^2 e^{2\pi i k \cdot (y-z)} \frac{(e^{2\pi i t k \cdot \alpha} - 1)^2}{4\pi^2 (k \cdot \alpha)^2}. \quad (4.25)$$

The first sum is 1 as in the proof of Theorem 1, while the small divisors in the second sum may be controlled with the truncated Diophantine condition. This yields

$$t^2 \left(1 - \frac{\delta K}{R^n}\right) \leq \frac{1}{C^2 \pi^2} \sum_{\|k\| \leq N} \|k\|^{2s} |u_k|^2. \quad (4.26)$$

As before in the estimates following (4.12), the right-hand side is estimated as

$$< \frac{\|V_*\|_\Delta^2}{C^2 \pi^2 R^{2s+n}}. \quad (4.27)$$

We now take

$$T^2 = \frac{4\|V_*\|_\Delta^2}{C^2 \pi^2 R^{2s+n} (1 - \delta K R^{-n})}, \quad (4.28)$$

and the proof concludes by way of arguments similar to those ending the proof of Theorem 1. $\square$

We now proceed to the proof of Theorem 2. Lemma 4.3 has been stated so that its proof closely parallels the proof of Theorem 1, but it would be more illuminating if it explicitly displayed the dependence of the ergodization time T on the cutoff N . Also, we know that for a given $R < 1$, there is a critical cutoff N_{crit} such that if $\alpha \cdot k = 0$ for some k with $\|k\| < N_{\text{crit}}$, then rectilinear orbits with direction vector α will not ergodize $\mathbf{T}^n$ at all.

Lemma 4.3 provides an estimate of both N_{crit} and the dependence of T on N (and so establishes Theorem 2) as follows. We simply solve (4.21) for δ :

$$\delta = 2\|V_*\|_{L^2} (R(N - \sqrt{n}/2))^{-(s-n/2)}, \quad (4.29)$$

in which case the condition $\delta < R^n/K$ ensuring that T is finite (cf. Eq. (4.27)) becomes

$$N > N_0 \equiv (2\|V_*\|_{L^2K})^{\frac{1}{s-n/2}} R^{-\left(\frac{s+n/2}{s-n/2}\right)} + \sqrt{n}/2. \quad (4.30)$$

(Note: when the taxicab norm is used in (3.5), the use of $|N| \leq \sqrt{n}\|N\|$ leads to (3.8) rather than (4.28) here.) We thus have the estimate $N_{\text{crit}} \leq N_0$. Theorem 2 follows immediately by identifying κ_o with $(2\|V_*\|_{L^2K})^{\frac{1}{s-n/2}}$ in (4.30) and by substituting (4.29) into (4.22) and simplifying.

APPENDIX: TEST FUNCTIONS AND TECHNICAL ESTIMATES

This appendix gathers together technical results on the test functions used in the proofs of the previous section. These results are standard, but are provided here for convenience.

A.1 Notation and Terminology for Functions on $\mathbf{R}^n$ and $\mathbf{T}^n$

At the risk of seeming unnecessarily pedantic, let us first digress about the relationships between functions on $\mathbf{T}^n$, their periodic lifts on $\mathbf{R}^n$, and what we will call their “finite lifts” on $\mathbf{R}^n$, which will be needed shortly.

Using the notation of Section 2, each function $f : \mathbf{T}^n \rightarrow \mathbf{R}$ gives rise to its lift $\tilde{f} : \mathbf{R}^n \rightarrow \mathbf{R}$ in the standard way via $\tilde{f} = f \circ p$. The lift of a function on $\mathbf{T}^n$ is of course unit-periodic in each coordinate of $\mathbf{R}^n$. Conversely, each multiply-unit-periodic function $g : \mathbf{R}^n \rightarrow \mathbf{R}$ generates a corresponding function g_T on $\mathbf{T}^n$ by means of $g_T = g \circ (p|_X)^{-1}$. Henceforth, we shall no longer distinguish between functions on $\mathbf{T}^n$ and multiply-unit-periodic functions on $\mathbf{R}^n$.

Next, for $u : \mathbf{T}^n \rightarrow \mathbf{R}$ (i.e., for u a multiply-unit-periodic function on $\mathbf{R}^n$), we define $u_* : \mathbf{R}^n \rightarrow \mathbf{R}$, the *finite lift* of u , as:

$$u_*(x) = \begin{cases} u(x), & \text{if } x \in \bar{\chi} \\ 0, & \text{otherwise} \end{cases}, \quad (A.1)$$

which clearly has support in the fundamental cube $\bar{\chi} = [-1/2, 1/2]^n$.

This formalism will be useful to us because a multiply-periodic function $u \in L^2(\mathbf{T}^n)$ is expressed by means of its Fourier series

$$u(\theta) = \sum_{k \in \mathbf{Z}^n} u_k e^{2\pi i k \cdot \theta}, \quad (A.2)$$

while its finite lift $u_* \in L^2(\bar{\chi})$ is given by the integral:

$$u_*(x) = \int_{\mathbf{R}^n} \hat{u}_*(\xi) e^{2\pi i \xi \cdot x} d\xi, \quad (A.3)$$

where the Fourier transform $\hat{u}_*$ of u_* is given by

$$\hat{u}_*(\xi) = \int_{\mathbf{R}^n} u_*(x) e^{-2\pi i \xi \cdot x} dx = \int_{\bar{\chi}} u(x) e^{-2\pi i \xi \cdot x} dx, \quad (A.4)$$

and the relation between the Fourier coefficients of u and the Fourier transform of u_* is

$$u_k = \int_{\mathbf{T}^n} u(\theta) e^{-2\pi i k \cdot \theta} d\theta = \int_{\bar{\chi}} u(\theta) e^{-2\pi i k \cdot \theta} d\theta = \hat{u}_*(k). \quad (A.5)$$

We will shortly use the equality $u_k = \hat{u}_*(k)$ together with the simple scaling properties of the Fourier transform to prove a key estimate.

A.2 The Sobolev norm $\|\cdot\|_\Delta$

For $0 < R \leq 1$, we now consider $H_0^s(B_{R/2}(0))$, the L^2 -Sobolev space of order $s > n/2$ on the open ball $B_{R/2}(0)$ of diameter R centered on 0 in $\mathbf{R}^n$. We will use asterisk subscripts to serve as a reminder that every $u_* \in H_0^s(B_{R/2}(0))$ is the finite lift of the corresponding function u on $\mathbf{T}^n$. $H_0^s(B_{R/2}(0))$ is a Hilbert space under the usual inner product compatible with the Sobolev norm, but the point of introducing this space here is to make use of the equivalent norm $\|\cdot\|_\Delta$ defined by

$$\|u_*\|_\Delta = \left(\int_{\mathbf{R}^n} \|\xi\|^{2s} |\hat{u}_*(\xi)|^2 d\xi \right)^{1/2} = (2\pi)^{-s} \|(-\Delta)^{s/2} u_*\|_{L^2} \quad (A.6)$$

for $u_* \in H_0^s(B_{R/2}(0))$. (The operator $-\Delta$ is defined below in (A.9).) In fact, given u_* , the same norm may be defined for the corresponding periodic function u by setting

$$\|u\|_\Delta = \left(\sum_{k \in \mathbf{Z}^n} \|k\|^{2s} |u_k|^2 \right)^{1/2} \quad (A.7)$$

and noting that this is consistent since

$$\|u_*\|_\Delta = \|u\|_\Delta. \quad (A.8)$$

This last equality is easily seen by applying Parseval's identity twice (once for series, once for transforms) to $(2\pi)^{-s}(-\Delta)^{s/2} u_*$ and $(2\pi)^{-s}(-\Delta)^{s/2} u$, where non-integral

powers of the self-adjoint positive operator $-\Delta$ (the negative Laplacian) have been defined for $0 \leq a \leq s/2$ by

$$(-\Delta)^a u_*(x) = (2\pi)^{2a} \int_{\mathbf{R}^n} \|\xi\|^{2a} \widehat{u}_*(\xi) e^{2\pi i \xi \cdot x} d\xi \in L^2(\bar{\chi}), \quad \text{and} \quad (A.9)$$

$$(-\Delta)^a u(\theta) = (2\pi)^{2a} \sum_{k \in \mathbf{Z}^n} \|k\|^{2a} u_k e^{2\pi i k \cdot \theta} \in L^2(\mathbf{T}^n), \quad (A.10)$$

and clearly $\|(-\Delta)^a u_*\|_{L^2(\bar{\chi})} = \|(-\Delta)^a u\|_{L^2(\mathbf{T}^n)}$.

The finite lifts were introduced to facilitate proofs of the scaling relationships for the test functions to be described below. These scaling relationships are naturally derived in $\mathbf{R}^n$, where Fourier transforms of finite lifts depend continuously on the parameter ξ , rather than in $\mathbf{T}^n$, where Fourier coefficients depend on discrete indices k .

A.3 The Test Functions

Consider the subset $\mathcal{A}_{R/2}^{s*}$ of $H_0^s(B_{R/2}(0))$ consisting of functions u_* satisfying

- (i) $u_* \geq 0$
- (ii) $\int_{\bar{\chi}} u_* = 1$.

We define $\mathcal{A}_{R/2}^s$, the set of test functions $u : \mathbf{T}^n \rightarrow \mathbf{R}$ used in the proofs of Theorems 1 and 2, by $u \in \mathcal{A}_{R/2}^s \iff u_* \in \mathcal{A}_{R/2}^{s*}$. The proofs of the scaling relationships below will be carried out in $\mathcal{A}_{R/2}^{s*}$.

The Hilbert space structure of $H_0^s(B_{R/2}(0))$ affords an easy way of seeing that $\mathcal{A}_{R/2}^{s*}$ has a unique element U_* minimizing $\|\cdot\|_\Delta$. In fact, $\mathcal{A}_{R/2}^{s*}$ may be characterized as the intersection of the cone of nonnegative functions with the affine translation, by U_* , of the linear subspace $\mathcal{O}$ of $H_0^s(B_{R/2}(0))$ consisting of functions w_* with $\int_{\bar{\chi}} w_* = 0$; where U_* is the unique element of $\mathcal{A}_{R/2}^{s*}$ satisfying $\|U_*\|_\Delta = \inf_{\mathcal{A}_{R/2}^{s*}} \|u_*\|_\Delta$. In other words, U_* is the unique element of $\mathcal{A}_{R/2}^{s*}$ lying in $\mathcal{O}^\perp$.

A.4 Technical Lemmas

We may now state and prove the lemmas needed in the proofs of Theorems 1 and 2. The first lemma concerns the scaling relationships between U_* (the function with support in $\bar{B}_{R/2}(0)$ which minimizes $\|\cdot\|_\Delta$) and V_* (the minimizing function with support in $\bar{B}_{1/2}(0)$); the second concerns the rates of decrease for the Fourier transform of V_* and the Fourier series of U .

Lemma A.1. Let $U_* \in \mathcal{A}_{R/2}^{s*}$ and $V_* \in \mathcal{A}_{1/2}^{s*}$ be the unique elements satisfying

$$\|U_*\|_\Delta = \min_{u_* \in \mathcal{A}_{R/2}^{s*}} \|u_*\|_\Delta \quad \text{and} \quad \|V_*\|_\Delta = \min_{v_* \in \mathcal{A}_{1/2}^{s*}} \|v_*\|_\Delta.$$

Then

- (i) $\|U_*\|_\Delta = R^{-s-n/2} \|V_*\|_\Delta$,
- (ii) $U_*(x) = R^{-n} V_*(x/R)$, where $x/R = (x_1/R, \dots, x_n/R)$, and
- (iii) $\widehat{U}_*(\xi) = \widehat{V}_*(R\xi)$.

PROOF. It is clear that every element $u_* \in \mathcal{A}_{R/2}^{s*}$ may be written in terms of exactly one element $v_* \in \mathcal{A}_{1/2}^{s*}$, and vice versa, via the relations $u_*(x) = R^{-n} v_*(x/R)$ and $v_*(x) = R^n u_*(Rx)$. Therefore, using (A.6), we have

$$\begin{aligned} \|U_*\|_\Delta &= \min_{u_* \in \mathcal{A}_{R/2}^{s*}} \|u_*(x)\|_\Delta = \min_{v_* \in \mathcal{A}_{1/2}^{s*}} \|R^{-n} v_*(x/R)\|_\Delta \\ &= R^{-n} \min_{v_* \in \mathcal{A}_{1/2}^{s*}} (2\pi)^{-s} \left(\int_{\mathbf{R}^n} |(-\Delta)^{s/2} v_*(x/R)|^2 dx \right)^{1/2} \\ &= R^{-s-n} \min_{v_* \in \mathcal{A}_{1/2}^{s*}} (2\pi)^{-s} \left(\int_{\mathbf{R}^n} |[(-\Delta)^{s/2} v_*](x/R)|^2 dx \right)^{1/2} \\ &= R^{-s-n/2} \min_{v_* \in \mathcal{A}_{1/2}^{s*}} (2\pi)^{-s} \left(\int_{\mathbf{R}^n} |(-\Delta)^{s/2} v_*(y)|^2 dy \right)^{1/2} = R^{-s-n/2} \|V_*\|_\Delta, \quad (A.11) \end{aligned}$$

from which (i) follows; (ii) follows in turn by the uniqueness of U_* and V_* .

The proof of (iii) is immediate using the definitions of $\widehat{U}_*$ and $\widehat{V}_*$ together with a change of variable. $\square$

Lemma A.2. Let U_* and V_* be defined as in the previous lemma. Then

- (i) $|\widehat{V}_*(\xi)| \leq K_1 \|\xi\|^{-s}$
- (ii) for $M > 1/R$, the tail series

$$\sum_{\substack{k \in \mathbf{Z}^n \\ \|k\| \geq M + \sqrt{n}/2}} |U_k|^2 \leq \frac{K^2}{R^{2s} M^{2s-n}},$$

where $K^2 = K_1^2 K_2^2 K_3^2$, and

$$K_1 = n^{s/2} (2\pi)^{-s} \max_{|\beta|=s} \|D^\beta V_*\|_{L^1},$$

$$K_2 = (1 + \sqrt{n}/2)^s, \text{ and}$$

$$K_3^2 = 2\pi^{n/2}(2s - n)/\Gamma(n/2).$$

PROOF. Part (i) is the familiar formula relating the rate of decrease of the Fourier transform (or Fourier coefficients) of a function to its differentiability; it is effected by integrating the transform's defining formula (A.4) by parts s times.

Part (ii) is proved as follows:

$$\begin{aligned} \sum_{\substack{k \in \mathbf{Z}^n \\ \|k\| \geq M + \sqrt{n}/2}} |U_k|^2 &= \sum_{\substack{k \in \mathbf{Z}^n \\ \|k\| \geq M + \sqrt{n}/2}} |\widehat{U}_*(k)|^2 \\ &= \sum_{\substack{k \in \mathbf{Z}^n \\ \|k\| \geq M + \sqrt{n}/2}} |\widehat{V}_*(Rk)|^2 \leq \frac{K_1^2}{R^{2s}} \sum_{\substack{k \in \mathbf{Z}^n \\ \|k\| \geq M + \sqrt{n}/2}} \frac{1}{\|k\|^{2s}}, \end{aligned} \quad (\text{A.12})$$

where part (iii) of Lemma A.1 and part (i) of the present lemma were used to obtain the last two expressions. It is easily verified that, with $K_2 = (1 + \sqrt{n}/2)^s$, the function $K_2 \|\xi\|^{-s}$ majorizes $\|k\|^{-s}$ on every closed unit n -cube $\bar{\chi}_k = \{\xi \in \mathbf{R}^n \mid |\xi_i - k_i| \leq 1/2, i = 1, \dots, n\}$ centered on the integer vector $k \in \mathbf{Z}^n$. Moreover, the set $\{\bigcup \bar{\chi}_k \mid k \in \mathbf{Z}^n \text{ with } \|k\| \geq M + \sqrt{n}/2\}$ is contained in $\{\xi \in \mathbf{R}^n \mid \|\xi\| \geq M\}$ ($\sqrt{n}/2$ is the distance from the center to a vertex of the unit n -cube), and it follows that the last expression of A.12 is bounded by

$$\frac{K_1^2 K_2^2}{R^{2s}} \int_{\|\xi\| \geq M} \frac{d\xi}{\|\xi\|^{2s}}. \quad (\text{A.13})$$

The integral may be evaluated as $\mathcal{S}_{n-1}(1) \cdot (2s - n)M^{n-2s}$, where $\mathcal{S}_{n-1}(1) = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the $n - 1$ sphere with unit radius. Combining this with the coefficient appearing in front of the integral shows that (A.13) equals $K_1^2 K_2^2 K_3^2 R^{-2s} M^{n-2s} = K^2 R^{-2s} M^{n-2s}$, as desired. $\square$

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