LAX PAIR FORMULATION FOR THE EULER EQUATION

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Abstract

A Lagrangian description is presented for the two and three dimensional flow of an ideal fluid . A Lax pair equation is obtained for the evolution of a quadratic form from which the Lagrangian velocity can be reconstructed. Inverse scattering in 2-D is discussed and a Hamiltonian structure is exhibited.

In this paper we use Lagrangian coordinates to present an evolution equation for a matrix denoted by Λ which governs the stretching and contraction of vectors "frozen" into the flow of an ideal fluid. This equation is exactly equivalent to Euler's equations for the motion of an ideal fluid evolving from an initial velocity $\underline{v}_0$. We show that the evolution equation for Λ can be written as a Lax pair. In contrast with Lax pair formulations for the Euler equations that follow directly from the Lie-Poisson structure, the operator L in the present paper associated with the spectral problem is nondegenerate (elliptic). Hence the formulation that we give is a suitable candidate for the method of inverse scattering. In the case of two dimensional flows that decay sufficiently rapidly at infinity, we outline the procedure by which Λ can be reconstructed via inverse scattering using so called " $\bar{\partial}$ techniques". It should, however, be noted that the applicability of inverse scattering to the solution of the unbounded problem, does not imply the integrability of the two dimensional Euler equation which remains an important open question.

Lagrangian Description

Let $\underline{v}(\underline{z}, t)$ be the Eulerian velocity field of a homogeneous incompressible inviscid fluid in a bounded domain Ω in 3-space. The Euler equation governing the flow is

$$\text{curl} \dot{\underline{v}} = \text{curl}(\underline{v} \times \text{curl} \underline{v}) , \quad \text{div} \underline{v} = 0 \quad (1)$$

$$\underline{v}(\underline{z}, 0) = \underline{v}_0(\underline{z}) , \quad (\underline{v}, \underline{n})|_{\partial\Omega} = 0 ,$$

where $(\dot{})$ denotes the derivative with respect to t .

There are different useful formulations of this problem, for example, the Hamiltonian formulation due to Arnold [1] or the more recent one due to V.I.Oseledecs [2]. Our intent here is to give an alternative formulation using a Lagrangian description of the flow. This leads to the construction of a so called Lax pair [3] which is probably of intrinsic significance for this problem and also for related problems of ideal hydrodynamics and magnetohydrodynamics. Although a Lagrangian formulation for fluid flow is a well known classical construction, we are not aware that the particular formulae that we will present have previously appeared in the literature.

We introduce the following Lagrangian coordinates $\underline{x}$: let $\underline{z}=g(\underline{x},t)$ be the solution of the ODE

$$\dot{\underline{z}}=\underline{v}(\underline{z},t), \quad \underline{z}(0)=\underline{x}. \quad (2)$$

The Jacobi matrix for this transformation is given by

$$g_{*}(\underline{x},t) = \partial \underline{z} / \partial \underline{x} \quad (3)$$

(In coordinate notation $g_{*i,j} = \partial z_i / \partial x_j$, $i=1,2,3$; $j=1,2,3$.)

We define a matrix $\Lambda(\underline{x},t) = g_{*}^T g_{*}$. The geometric interpretation of Λ is that the quadratic form $(\Lambda(\underline{x},t)\underline{\eta},\underline{\eta})$ gives the square of the "frozen" vector $\underline{\eta}$ at time t i.e. $(\Lambda(\underline{x},t)\underline{\eta}(0),\underline{\eta}(0))=(\underline{\eta}(t),\underline{\eta}(t))$. We note that Λ is used in nonlinear elasticity theory where it is called the inverse Cauchy-Green tensor (see, for example, Marsden and Hughes [4]).

The Lagrangian velocity $\underline{u}(\underline{x},t)$ is defined by the equality

$$\underline{u}(\underline{x},t)=g_{*}^{-1}\underline{v}(\underline{z},t). \quad (4)$$

It can be easily shown [5] that the following operator identity holds

$$g_{*}^{-1} \text{curl } g_{*}(\cdot) = \text{curl } \Lambda(\cdot), \quad (5)$$

where g_* is considered as the shift operator along the trajectories of the ODE (2), while Λ on the RHS of (5) is matrix multiplication by $\Lambda(\underline{x}, t)$. It follows that

$$\text{curl}(\underline{z}, t) = g_* \text{curl} \Lambda \underline{u}(\underline{x}, t). \quad (6)$$

Substituting (6) into Euler's equation given by (1), we obtain

$$\partial/\partial t \text{curl} \Lambda \underline{u} = 0, \quad (7)$$

hence,

$$\text{curl} \Lambda(\underline{x}, t) \underline{u}(\underline{x}, t) = \text{curl} \underline{v}_0(\underline{z}). \quad (8)$$

This is equivalent to Kelvin's theorem for conservation of vorticity. Equation (8) implies that

$$\Lambda \underline{u} = \underline{v}_0 + \nabla p \quad (9)$$

for some scalar p .

We now deduce the evolution equation for $\Lambda(\underline{x}, t)$. Let $\eta(\underline{z}, t)$ satisfy the advection equation

$$\dot{\eta}(\underline{z}, t) = -(\underline{v}(\underline{z}, t), \nabla) \eta(\underline{z}, t) + (\eta(\underline{z}, t), \nabla) \underline{v}(\underline{z}, t) \equiv \{\underline{v}, \eta\} \quad (10)$$

It follows from the definition of Λ that

$$\begin{aligned} d/dt \int (\Lambda(\underline{x}, t) \eta(\underline{x}, 0), \eta(\underline{x}, 0)) d^3 x &= d/dt \int (\eta(\underline{z}, t), \eta(\underline{z}, t)) d^3 z \\ &= \int (\{\underline{v}, \eta\}, \eta) d^3 z + \int (\eta, \{\underline{v}, \eta\}) d^3 z \\ &= \int (g_* \{\underline{u}, \eta(0)\}, g_* \eta(0)) + (g_* \eta(0), g_* \{\underline{u}, \eta(0)\}) d^3 x \\ &= \int (\Lambda \{\underline{u}, \eta(0)\}, \eta(0)) + (\eta(0), \Lambda \{\underline{u}, \eta(0)\}) d^3 x \\ &= \int ((\underline{u}, \nabla) \Lambda) \eta(0), \eta(0)) + (\Lambda \partial \underline{u} / \partial \underline{x} \eta(0), \eta(0)) + ((\partial \underline{u} / \partial \underline{x})^T \Lambda \eta(0), \eta(0)) d^3 x, \end{aligned} \quad (11)$$

where the final equality uses the solenoidality of $\underline{u}$. Hence the evolution equation for Λ is as follows

$$\dot{\Lambda} = (\underline{u}, \nabla) \Lambda + \Lambda (\partial \underline{u} / \partial \underline{x}) + (\partial \underline{u} / \partial \underline{x})^T \Lambda, \quad \Lambda(\underline{x}, 0) = I. \quad (12)$$

Equation (12) is of purely geometric nature: it describes the "advection" of the quadratic form by the flow. However equation (9) specifies the flow of the fluid. It should be stressed that one could use (9) to express the solenoidal vector $\underline{u}$ as the image

of $\underline{v}_0$ under some pseudodifferential operator depending on nothing but Λ . Inverting Λ and taking the divergence of both sides of (9) we have

$$\operatorname{div} (\Lambda^{-1} \underline{v}_0) + \operatorname{div} (\Lambda^{-1} \nabla p) = 0. \quad (13)$$

Solving this second order equation under the boundary condition

$$(\Lambda^{-1} \underline{v}_0 + \Lambda^{-1} \nabla p, \underline{n})|_{\partial\Omega} = 0, \quad (14)$$

we obtain ∇p and hence $\underline{u}$ which we write as $\hat{\Lambda}^{-1} \underline{v}_0$. It is clear that the above description is also valid for periodic boundary conditions and for the free space problem with decay at infinity. The pseudodifferential operator $\hat{\Lambda}^{-1}$ of zero order corresponds to the main symbol

$$\underline{\zeta} \longrightarrow -(\underline{\xi} \times (\Lambda^{-1}(\underline{x}, t) \underline{\xi} \times \Lambda^{-1}(\underline{x}, t) \underline{\zeta})) / (\Lambda^{-1}(\underline{x}, t) \underline{\xi}, \underline{\xi}).$$

The Fourier variable $\underline{\zeta}$ is perpendicular to $\underline{\xi}$ because vectors are solenoidal. Using this pseudodifferential operator of zero order we obtain the evolution equation for the matrix $\Lambda(\underline{x}, t)$ in the form

$$\dot{\Lambda} = (\hat{\Lambda}^{-1} \underline{v}_0, \nabla) \Lambda + \Lambda \partial / \partial \underline{x} (\hat{\Lambda}^{-1} \underline{v}_0) + (\partial / \partial \underline{x} (\hat{\Lambda}^{-1} \underline{v}_0))^T \Lambda. \quad (15)$$

We note that $\Lambda(\underline{x}, t)$ completely describes stretching and contraction in the ideal fluid. The Lagrangian velocity vector could in principle be determined by solving (15) and then by using (9). The symmetric matrix Λ has determinant 1 because of the volume preserving property. Furthermore the Riemannian metric $(\Lambda(\underline{x}, t) d\underline{x}, d\underline{x})$ is flat, i.e. all the components of Riemannian curvature vanish, implying three conditions on the components of Λ . Hence the description of the flow given by equation (15) involves the same number of independent functions as the Eulerian description given in (1). We observe that although equation (12) for Λ is purely kinematic, equation (15) describes the dynamics of the flow as it evolves from a given initial velocity field $\underline{v}_0$.

Equation (15) for Λ is exactly equivalent to the Euler equations governing the flow of an ideal fluid whose initial velocity is $\underline{v}_0$.

The analogous Lagrangian description of the Navier-Stokes equation governing the flow of a viscous fluid is as follows :

$$\partial/\partial t \text{ curl}(\Lambda \underline{u}) = -\nu (\text{curl } \Lambda)^3 \underline{u}, \quad (\nabla, \underline{u})=0; \quad (16)$$

$$\underline{u}(\underline{x}, 0) = \underline{v}_0(\underline{x}),$$

where ν is the kinematic viscosity. The system (12), (16) governs both the anisotropic diffusion of vorticity and the evolution of the anisotropy described by the matrix Λ .

A Lax Pair for the Euler Equation

It is of interest to observe that the evolution equation (15) can be written in terms of a Lax pair. Manipulations similar to those used in obtaining equation (11) lead to the following formulation of (15) :

$$\partial/\partial t \text{ curl} \Lambda(.) = [\text{curl} \Lambda(.), \{ \Lambda^{-1} \underline{v}_0, (.) \}], \quad (17)$$

where the Poisson bracket $\{.,.\}$ is defined in (10) and $[.,.]$ denotes the commutator of the two operators of the Lax pair acting on solenoidal vector fields. The existence of a Lax pair is associated with the inverse scattering method for solving certain classes of nonlinear initial value problems [6]. In the following section we will discuss briefly results concerning inverse scattering for two dimensional flows. As the referee observes, the kinematic advection equation (12) for Λ is inherited from the Lie-Poisson structure of the fluid equations. This structure has been exploited by many authors including Holm and Kupershmidt [7] and Marsden et al [8]. However, to our knowledge,

inverse scattering techniques could not be applied to a highly degenerate differential operator like a Poisson bracket. In contrast we will indicate how, in the two dimensional case the elliptic (nondegenerate) operator associated with equation (17) has the requisite analytic properties for inverse scattering.

For two dimensional flows equation (17) reduces to an equation, involving scalar operators which is given as follows:

$$\partial/\partial t \operatorname{div} \Lambda^{-1} \nabla(.) = [\operatorname{div} \Lambda^{-1} \nabla(.), -(\Lambda^{-1} \underline{v}_0, \nabla)(.)]. \quad (18)$$

The zero curvature condition in two dimensions is given by

$$\partial^2 \Lambda_{12} / \partial x_1 \partial x_2 - 1/2 \partial^2 \Lambda_{11} / \partial x_2 \partial x_2 - 1/2 \partial^2 \Lambda_{22} / \partial x_1 \partial x_1 = 0. \quad (19)$$

Equation (18) can be generalized to the case of three dimensional flows which are independent of one spatial coordinate which we will denote by x_3 . We introduce the following notation : let $\Lambda_{\perp}$ be the restriction of the matrix Λ to the (x_1, x_2) -plane, $w(x_1, x_2) = z_3(x_1, x_2, x_3) - x_3$, $\underline{u} = \underline{1}_3 \times \nabla \phi + \underline{1}_3 u_3$. The Lax pair equation becomes

$$\partial/\partial t \begin{bmatrix} 0 & -I \\ \operatorname{div} \Lambda_{\perp}^{-1} \nabla & \{., w\} \end{bmatrix} = \left[\begin{bmatrix} 0 & -I \\ \operatorname{div} \Lambda_{\perp}^{-1} \nabla & \{., w\} \end{bmatrix}, \begin{bmatrix} -(\underline{u}, \nabla) & 0 \\ -\{u_3, .\} & \{., \phi\} \end{bmatrix} \right],$$

$$u_3 = u_3(0) \text{ and } \operatorname{div} \Lambda_{\perp}^{-1} \nabla \phi + \{u_3, w\} = \Delta \phi(0). \quad (20)$$

This defines the parameters $\Lambda_{\perp}$ and w subject to initial conditions $\Lambda_{\perp} = I$, $w(0) = 0$.

The preceding discussion for an ideal fluid generalizes to give a Lagrangian description for the flow of an electrically conducting fluid. Let $\underline{H}$ be the Eulerian magnetic field vector satisfying the equation

$$\dot{\underline{H}} = \operatorname{curl} (\underline{v} \times \underline{H}), \quad \underline{H}(\underline{z}, 0) = \underline{H}_0. \quad (21)$$

We note that $\underline{H} = g_{\underline{x}} \underline{H}_0$ is the solution of (21), hence $\underline{H}_0$ is the Lagrangian magnetic field. The magnetic analogue of equation (7)

is

$$\partial/\partial t \text{ curl } \Lambda \underline{u} = \text{curl}((\text{curl } \Lambda \underline{H}_0) \times \underline{H}_0). \quad (22)$$

By integrating (22) we obtain

$$\Lambda \underline{u} = \underline{v}_0 + (\text{curl} \int_0^t \Lambda(\tau) d\tau \underline{H}_0) \times \underline{H}_0 + \nabla p. \quad (23)$$

Equations (12) and (23) describe the Lagrangian velocity of a conducting fluid via the matrix Λ . Equations (12) and (22) admit the following Lax pair representation :

$$\dot{L} = [L, M], \quad (24)$$

where

$$L = \begin{bmatrix} \text{curl } \Lambda & 0 & 0 \\ 0 & -\{\underline{H}_0, \cdot\} & \{\text{curl } \Lambda \underline{u}, \cdot\} \\ 0 & 0 & -\{\underline{H}_0, \cdot\} \end{bmatrix}, \quad M = \begin{bmatrix} \{\underline{u}, \cdot\} & 0 & 0 \\ 0 & 0 & \{\text{curl } \Lambda \underline{H}_0, \cdot\} \\ 0 & 0 & 0 \end{bmatrix}$$

The variables in this representation are Λ and $\Lambda \underline{u}$.

Inverse Scattering for 2-D Flow of an Ideal Fluid

We now briefly discuss inverse scattering for two dimensional fluid flows in the context of the Lax pair equation (18). Details of these results will be given in the forthcoming paper. We apply inverse scattering techniques to the 2-D elliptic operator

$$L \equiv -\text{div } \Lambda^{-1} \nabla(\cdot)$$

using the $\bar{\partial}$ philosophy of Beals and Coifman [9]. In analogy with the 2-D Schrodinger equation, we study the spectral problem given by

$$-\text{div}(\Lambda^{-1} \nabla \psi) = s^2 \psi \quad (25)$$

where the spectral parameter s is a two dimensional complex vector. Under the condition that $(\Lambda^{-1}(\underline{x}) - I)$ decays at infinity as $|\underline{x}|^{-3}$, we consider the eigenvalue problem

$$-\nabla^2 \psi - \text{div}(\Lambda^{-1} - I) \nabla \psi = s^2 \psi \quad (26)$$

as a perturbation of the same problem for the Laplacian. Following the notation of Beals and Coifman [9] we define

$$p(D) = D^2 = -\nabla^2, \quad q(\underline{x}, D) = \operatorname{div} (\Lambda^{-1} - I) \nabla. \quad (27)$$

We seek eigenfunctions for (26) in the form

$$\psi = m(\underline{x}, \underline{s}) e^{i(\underline{x}, \underline{s})} \quad (28)$$

where $m \rightarrow 1$ as $|\underline{x}| \rightarrow \infty$.

From (26)-(28) we obtain the equation for $m(\underline{x}, \underline{s})$ as

$$p_{\underline{s}}(D) m(\underline{x}, \underline{s}) = q(\underline{x}, D + \underline{s}) m(\underline{x}, \underline{s}) \quad (29)$$

where $p_{\underline{s}}(\xi) = p(\xi + \underline{s}) - p(\underline{s})$.

Let $G_{\underline{s}}$ be the Green's function for the operator $p_{\underline{s}}(D)$, i.e.

$$G_{\underline{s}}(\underline{x} - \underline{y}) = (2\pi)^{-2} \int e^{i(\underline{\xi}, \underline{x} - \underline{y})} (\xi^2 + 2(\underline{\xi}, \underline{s}) + s^2)^{-1} d^2 \xi. \quad (30)$$

From (29) the integral equation for $m(\underline{x}, \underline{s})$ can be written as follows

$$m = 1 + G_{\underline{s}} q(\underline{x}, D + \underline{s}) m \equiv 1 + K_{\underline{s}} m. \quad (31)$$

We follow arguments that have been successfully used in applying inverse scattering to other 2-D problems. For example, in the problem of reconstructing the 2-D Schrödinger operator for a quantum particle in a magnetic field, it is sufficient to consider scattering data only for a fixed energy level. Loosely speaking, this approach is motivated by the fact that because s involves four real parameters, there is a redundancy of information in seeking to reconstruct a function of two real variables from scattering data for all values of s . The approach also has a deeper explanation following from the complex analytic properties of the scattering data [9], [10], [11].

We consider equations (26)-(31) for $\underline{s}$ belonging to the Riemann surface

$$V = \{\underline{s} \in \mathbb{C}^2 / p(\underline{s}) = 0\}. \quad (32)$$

In other words we are considering the zero energy level for (nonphysical) scattering data. This means that we consider only a fixed value of the spectral parameter in the Lax pair representation, namely zero. Similar considerations are applicable to any fixed energy level $p(\underline{s})=c$, where c is an arbitrary constant. Beals and Coifman [9] give arguments for the Schrödinger operator (i.e. (26)-(32) with $q(\underline{x}, D+\underline{s})$ replaced by $q(\underline{x})$) to show that the potential can be reconstructed from the scattering data. These arguments are also applicable to the problem under discussion with the decaying quantity $(\Lambda^{-1} - I)$ playing a role analogous to that of the potential. In order to justify the formal approach used to obtain the inverse scattering transform, it is necessary to check that the eigenfunctions of (26) have the appropriate asymptotic behaviour. Using the integral equation (31) together with the asymptotics of the Green's function (30), it can be shown that

$$m(\underline{x}, \underline{s}) \sim 1 + O(|\underline{s}|^{-1/4}) \text{ as } \underline{s} \rightarrow \infty \text{ along } V. \quad (33)$$

To apply the $\bar{\partial}$ approach of Beals and Coifman [9] we need the $\bar{\partial}$ equation together with a Liouville's theorem for the singular Riemannian surface V .

The time evolution of the asymptotics of the eigenfunctions and the "potential" Λ can be easily determined from equation (9) and (12) to give the expressions

$$\begin{aligned} m(\underline{x}, \underline{s}) &\sim I + it(\underline{s}, \underline{v}_0(\underline{x})) + O(|\underline{x}|^{-3}), \\ \Lambda(\underline{x}, t) &\sim I + t(\partial \underline{v}_0 / \partial \underline{x} + (\partial \underline{v}_0 / \partial \underline{x})^T) + O(|\underline{x}|^{-4}). \end{aligned}$$

We note that the constructive procedure for obtaining $m(\underline{x}, \underline{s})$ requires the imposition of the condition that $(\Lambda^{-1} - I)(1 + |\underline{x}|^3)$ is sufficiently small. Thus the inverse scattering method that we have

outlined gives a recipe for constructing the solution only for small time t .

Hamiltonian Structure

It is of interest to define a Hamiltonian structure corresponding to coordinates $\Lambda(\underline{x})$ describing the flow. Let X denote the manifold consisting of all the Riemannian metrics in Ω connected with the standard Euclidean metric by some volume preserving diffeomorphism g . The tangent space to X at the point Λ_0 consists of all the vectors of the type

$$a_{\underline{\xi}} = (\underline{\xi}, \nabla) \Lambda_0 + (\partial \underline{\xi} / \partial \underline{x})^T \Lambda_0 + \Lambda_0 (\partial \underline{\xi} / \partial \underline{x}) . \quad (34)$$

We note that the solenoidal vector field $\underline{\xi}$ tangent to the boundary $\partial\Omega$ can be reconstructed from $a_{\underline{\xi}}$ up to a finite number of degrees of freedom. There is no freedom at all except in the case where the domain Ω admits some rigid body movement. Since there is a natural action of the group $\text{SDiff}\Omega$ on the manifold X , it is sufficient to check the previous statement in the trivial case when $\Lambda_0 = I$. For simplicity we consider the general rather than the exceptional case. The Lagrangian function is defined as follows

$$\mathcal{L}(\dot{\Lambda}, \Lambda) = (\Lambda \dot{\underline{\xi}}, \dot{\underline{\xi}}), \text{ where } \dot{\Lambda} = a_{\dot{\underline{\xi}}}. \quad (35)$$

This Lagrangian function is invariant under the action of $\text{SDiff}\Omega$ and the geodesics of (35) satisfy the Euler evolution equation (15) with the initial conditions

$$\Lambda(0) = I, \quad \dot{\Lambda}(0) = \partial \underline{v}_0 / \partial \underline{x} + (\partial \underline{v}_0 / \partial \underline{x})^T. \quad (36)$$

In comparison with Arnold's [1],[12] description of the Hamiltonian structure on the orbit K of the coadjoint

representation of $S\text{Diff}\Omega$, the map $\underline{\xi} \rightarrow a_{\underline{\xi}}$ has no, or almost no, null space. The null space in the Arnold description depends on the vector field in a highly complex manner and this is responsible for the complicated geometry of the orbits. Lax pairs associated with the Hamiltonian structure on the orbit K are highly degenerate and hence unsuitable for inverse scattering. We have shown that Lagrangian rather than Eulerian coordinates are the natural coordinates for a Lax pair formulation for the Euler equations in which the operator associated with the spectral problem is nondegenerate (elliptic).

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