REALIZATION AND NASH IMPLEMENTATION:
TWO ASPECTS OF MECHANISM DESIGN

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Realization and Nash Implementation: Two Aspects of Mechanism Design^{*}

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ABSTRACT

In this paper we will show how a message process which "realizes" (or computes) a given social choice rule F can be used to construct a game which implements F in Nash equilibrium. Any efficient encoding of information that occurs in the message process causes a corresponding reduction in the size of the strategy space of the game which we will construct to implement F . Necessary and (stronger) sufficient conditions on the message process will be given for this construction.

1. Introduction

The purpose of this paper is to connect two distinct lines of research that lie within the field of mechanism design. These lines are clearly connected in Hurwicz's [4] original formulation of the design problem, but most of the subsequent research has fallen into either one of two distinct categories.

The first of these focuses on the following problem. In order to implement some objective in an economic setting, information must be communicated by the agents, either to each other, or to some governing body. Information is communicated by means of a "message process", whose language is given by the elements of the "message space" (these terms will be defined precisely in Section 2). The first of these two approaches seeks to determine how "large" of a message space is needed to implement the objective, and also to explore the alternative message processes which can be used to implement the objective. As in team decision theory, this line of research is not concerned with the incentives of the individual agents. For the most part, they are regarded as automata in this approach, and attention is focused on determining the information which must be obtained from them in order to compute the objective.

This line of research is typified by the work of Hurwicz, Reiter, and Saari [5], and of Saari [13], wherein algorithms are devised for designing message processes which collect sufficient information to compute a given objective function. The highest achievement of this approach is the proof that the price mechanism is "informationally efficient" among all message processes which compute Pareto-optimal allocations for an economy (see Hurwicz [2], Mount and Reiter [10], and Osana [11]).

The second line of research that we will be discussing concerns the issue of incentives in economic mechanisms. This problem is currently being studied by many economists, in many different ways. In this paper, our attention will

be focused on the Nash solution concept, and our approach stems most naturally from the work of Hurwicz and Schmeidler [6], and Maskin [7,8].

The relationship between these two approaches to mechanism design was first explicitly studied in an abstract form by Reichelstein [12]. Briefly, this relationship can be characterized in the following manner. When some objective (or social choice correspondence) is implemented in Nash equilibrium by some game, the Nash equilibria of this game assume the role of the message space (as defined in Mount and Reiter [10]): the set of Nash equilibria in the strategy space represents the information which is actually used (in equilibrium) by the game to compute the objective. Alternatively, we might imagine that the message space has been "embedded" in a larger strategy space in such a way that it serves as the set of Nash equilibria of the game.

This suggests the approach that will be taken in this paper. We will assume that a message process which can be used to implement the objective is given, and our primary goal will be to determine when a "copy" of the message space can appear as the set of Nash equilibria (or at least a subset of this set) for some game which implements the objective in Nash equilibrium. We will derive a necessary condition which a message process must satisfy for this to be possible. A set of sufficient conditions for embedding a message space as a subset of the set of Nash equilibria of some larger game will also be discussed, and our proof that these conditions are sufficient will be constructive.

These results are important, because they will enable us to apply our knowledge of how to construct a message process which implements a given performance standard to the problem of designing a game which implements that performance standard in Nash equilibrium. Except for Maskin's [7] procedure for constructing a game with an extremely large strategy space, there do not exist any explicit procedures for designing games which implement a given performance standard. The problem of how to design message processes which implement a

given performance standard has been successfully approached, however, beginning with Mount and Reiter [10], and followed by the work of Hurwicz, Reiter, and Saari [5], Saari [13], and Williams [14]. Our work thus indicates how the growing literature on the problem of constructing message processes can be used to simplify the Nash implementation problem.

Before our results can be discussed more precisely, we must define several terms which we will be using in this paper. This will be the content of the next section. We will then summarize our results in Section 3.

2. Definitions and Notation.

A correspondence δ from set B to set C is a mapping from B to the set of subsets of C . For b in B , we use $\delta(b)$ to denote the subset which δ assigns to b . The correspondence δ is surjective if for any c in C , there exists an element b of B such that c is an element of $\delta(b)$. We will sometimes write $\delta(B) = C$ when δ is surjective.

We assume that there are n agents in the economy, and that the characteristics and information of the j^{th} agent is determined by a point in a set E_j . The Cartesian product $\prod_{i=1}^n E_i$ of these sets is the set of environments for the economy. We will usually use E to represent this product. Throughout most of this paper, we will not assume that the sets $E_1, E_2, \dots, E_n$ have any particular geometric structure.

A performance standard (or social choice rule) is a correspondence F from E into some set A , the set of alternatives.

Each element of E_j is presumed to define a complete preordering (as defined by Debreu [1, p. 7-8]) on A which determines the j^{th} agents' preferences over the elements of A . When the j^{th} agent is characterized by the point e_j in E_j , we will write " $a(>, e_j)b$ " when agent j with characteristics e_j either prefers outcome a of A to b of A , or else is

indifferent between these elements of A . Similarly, we will write " $a(>,e_j)b$ " when agent j strictly prefers a to b . We will use $L(a,e_j)$ to denote the lower contour set of the element a of A when the preferences are determined by the element e_j of E_j :

$$L(a,e_j) = \{b \in A | a(>,e_j)b\}.$$

A top-ranked alternative for an agent with characteristics e_j is an element a of A which satisfies $a(>,e_j)b$ for all b in A .

Each element e_j of E_j is presumed to define a complete preordering on A , but unlike much of the social choice literature, we do not make any assumption about how large of a subset of the set of all possible complete preorderings on A is actually determined as e_j varies over E_j . This aspect of the problem would depend upon which particular economic situation is being modeled.

We will adhere to Mount and Reiter's [10] formulation of message processes (which they refer to as "Hurwicz resource allocation processes"). Let M denote some set. A coordinate correspondence μ from E into M is a correspondence for which there exists correspondences $\mu_1, \mu_2, \dots, \mu_n$ that have the following properties:

- i) μ_j is a correspondence from E_j into M ;
- ii) $\mu((e_i)) = \bigcap_{i=1}^n \mu_i(e_i)$, for any point (e_i) in E .

A message process consists of a set M (the message space), a function h from M into A (the outcome function), and a surjective coordinate correspondence μ (the message correspondence) from E to M . For simplicity, we will use (μ, M, h) to represent this message process.

Our insistence that μ be a coordinate correspondence models the idea that each agent's messages are determined solely on the basis of his own information and preferences; to use Hurwicz's [3] term, μ "respects privacy". Since this interpretation is fully explained in Mount and Reiter [10], it will not be

discussed further here. Our assumption that μ is surjective is a matter of convenience: if μ were not surjective, then one would merely have to replace M with the set $\bigcup_{(e_i) \in E} \mu((e_i))$.

A message process (μ, M, h) realizes the performance standard F if

$$h \circ \mu((e_i)) \in F((e_i))$$

for all (e_i) in E , and (μ, M, h) fully realizes F if

$$h \circ \mu((e_i)) = F((e_i))$$

at any environment (e_i) . The term "realization" was introduced by Mount and Reiter [10] to distinguish this sort of implementation, which does not attempt to handle the problems that are posed by the issue of incentives.

We will now formally describe the second kind of implementation which will be studied in this paper. An n -person game on A consists of sets $S_1, S_2, \dots, S_n$ and a mapping g from the Cartesian product $\prod_{i=1}^n S_i$ into A . The mapping g will be referred to as the game form, the set S_j as the j^{th} agent's strategy space, and the product $\prod_{i=1}^n S_i$ as the strategy space of the game. We will usually use S to denote this product.

Given an element $(s_1, s_2, \dots, s_n)$ in S and a point s_j^* in S_j , we will use (s_j^*, s_{-j}) to denote the point $(s_1, \dots, s_{j-1}, s_j^*, s_{j+1}, \dots, s_n)$ obtained by changing the s_j -element of $(s_1, s_2, \dots, s_n)$ from s_j to s_j^* . The j^{th} agent's Nash equilibrium correspondence τ_j is defined on E_j with values in S . For e_j in E_j , the value of τ_j at e_j is given by the following formula:

$$\tau_j(e_j) \equiv \{(s_i) | g((s_j^*, s_{-j})) \in L((s_i), e_j), \forall s_j^* \in S_j\}.$$

The Nash equilibrium correspondence τ for the game is defined on E with values in S . It is defined by the formula $\tau((e_i)) = \bigcap_{i=1}^n \tau_i(e_i)$. Note that this correspondence is a coordinate correspondence.

Since our emphasis will be on the Nash solution concept, we will write (τ, S, g) to represent the game with strategy space S , game form g , and Nash equilibrium correspondence τ . The game (τ, S, g) implements the performance standard F in Nash equilibrium if the following conditions are satisfied:

- i) for any (e_i) in E , $\tau((e_i))$ is nonempty;
- ii) $g \circ \tau((e_i)) \in F((e_i))$ for all (e_i) in E .

Moreover, this game fully implements F in Nash equilibrium if g and τ also satisfy

$$g \circ \tau((e_i)) = F((e_i))$$

at any environment (e_i) .

"Implement" and "fully implement" will hereafter be reserved solely to describe implementation via the Nash equilibria of a game. The distinction between "implementation" and "full implementation" and between "realization" and "full realization" will not prove to be a major issue in this paper, so we will often state results using "(fully) realize" and "(fully) implement" to indicate that the result holds when either pair ("realize" and "implement", or "fully realize" and "fully implement") is used.

We will let $NE(g)$ denote the set of all elements of S which are Nash equilibria for at least one determination of the environment:

$$NE(g) \equiv \bigcup_{(e_i) \in E} \tau((e_i)) .$$

Let g^* denote the restriction of g to $NE(g)$. Note that if the game (τ, S, g) (fully) implements F , then the message process $(\tau, NE(g), g^*)$ (fully) realizes F . Roughly speaking, this paper is devoted to finding a converse to this statement. A more detailed discussion of the major issues of this paper and a summary of its principal results can be found in the next section.

3. Summary of Results

Given a message process (μ, M, h) which realizes the performance standard F , when can M serve as the set of Nash equilibria and μ as the Nash equilibrium correspondence of some "larger" game which implements F ? Most of this paper is actually devoted to determining when M can serve as a subset of the set of Nash equilibria of some game which implements F . This problem can be defined in a more precise manner if we use the notation which was developed in the preceding section.

Definition. Let (μ, M, h) be a message process which (fully) realizes the performance standard F , and let (τ, S, g) be a game which (fully) implements F in Nash equilibrium. The game (τ, S, g) contains (μ, M, h) if there exists an embedding of M into S such that the following conditions are satisfied when M is regarded as a subset of S :

$$i) \ M \text{ is a subset of } NE(g); \quad (1)$$

$$ii) \text{ for any } (e_i) \text{ in } E, \mu((e_i)) = \tau((e_i)) \cap M; \quad (2)$$

$$iii) \ g(m) = h(m) \text{ for any } m \text{ in } M. \quad (3)$$

Moreover, we will say that (τ, S, g) matches (μ, M, h) if these additional conditions hold under the embedding:

$$i)' \ M = NE(g); \quad (4)$$

$$ii)' \text{ for any } (e_i) \text{ in } E, \mu((e_i)) = \tau((e_i)). \quad (5)$$

At this stage, "embedding M into S " means simply, "to place M in a one-to-one correspondence with some subset of S ". In the case where both M and S have geometric structure (i.e., when they are topological spaces, or even manifolds), it would be desirable for the embedding to respect this structure. This difficulty will be discussed briefly below, and again later in the paper.

Also, note that (1) and (4) are really unnecessary. They follow from (2) and (5) (respectively), since we have assumed that μ is a surjective correspondence from E to M .

Most of the results in this paper concerns the question of whether or not there exists a game which contains a given message process. The question of the existence or nonexistence of game which matches a given message process is perhaps a more interesting question, but it is also a much harder question. It will be discussed, however, in Theorem 6.

We can now summarize our results. In Theorem 1, a necessary condition on (μ, M, h) will be derived for the existence of a game (τ, S, g) which contains (μ, M, h) . In Theorem 2, an equivalent form of this condition will be discussed. This version of the condition enables one to check μ and h directly to determine whether or not (μ, M, h) can serve as a subset of the set of Nash equilibria of some larger game. Theorem 2 thus isolates the properties which make a Nash equilibrium correspondence different from an ordinary coordinate correspondence. In Theorem 3, we will show how the conclusion of Theorem 2 is related to a property that the performance standard F must have if it can be implemented in Nash equilibrium -- monotonicity.

Slightly stronger sufficient conditions on (μ, M, h) for the existence of a game (τ, S, g) which contains (μ, M, h) will be given in Theorems 4, 5, and 7. Theorem 4 concerns the case when $n > 4$, while Theorem 5 is restricted to the three agent case. Theorem 7 concerns an alternative to one of the hypotheses of Theorems 4 and 5. All three of these theorems are proven with basically the same construction, which is outlined in the proof of Theorem 4. It will be clear from this construction process that the geometric structure of M is "preserved" when we identify it with a subset of the set of strategies of the game which we will construct: our space of strategies will be M^n , and M will be identified with the diagonal of this space.

In Theorem 6, we will discuss sufficient conditions for the existence of a game which matches a given message process. The proof of this theorem will also just be a modification of the construction process which is used to prove Theorems 4, 5, and 7.

The procedure that we will use for constructing games which have the desired properties is a generalization of the construction process that Maskin [7] developed to show that a performance standard with certain properties could be fully implemented in Nash equilibrium. The game which Maskin constructed has a strategy space which is potentially extremely large, depending upon the "size" of E and of A . Maskin recognizes that such a large set of strategies could be very inefficient because of the associated costs of communication and computation, and so he suggests that his procedure may only provide an a priori upper bound on the size of the set of strategies which would be needed by a game which implements the performance standard in Nash equilibrium.

In fact, Maskin's proof contains the germ of a procedure for constructing games for implementing the given performance standard that use strategy spaces which are potentially much smaller than his upper bound. A message process (μ, M, h) which realizes the performance standard may represent a significant condensation of all the possible information about an economy (as given by the space of environments) to some potentially much smaller "language" (which is determined by the space M); for instance, in the price mechanism, "normalized prices" and "net trades" represents the contraction of an immense amount of information (each agent's preferences, and his initial endowment) into a relatively small number of signals. The extent to which the information in E can be condensed has been clearly shown to depend upon the properties of the performance standard (see, for instance, the papers by Hurwicz, Reiter, and Saari[5] and Williams [14]). Our generalization of Maskin's technique will show that any condensation of information that is represented by the message space M

cause a corresponding reduction in the size of the set of strategies that is needed by a game which implements the performance standard in Nash equilibrium. Using our knowledge about the kinds of message processes which realize the given performance standard together with this procedure, it is thus sometimes possible to significantly decrease our estimate of the upper bound on the size of the strategy space which is needed for a Nash implementation of the given performance standard. For instance, when the message process that we start with uses a message space of dimension m , the game that we will construct has a strategy space of dimension nm . This will typically be quite a bit smaller than the strategy space which Maskin's game form uses!

As part of our discussion, we will correct a mistake in the original application of Maskin's construction process. We are referring to the result which was alluded to above -- Maskin's assertion [7, Thm. 5] that if there are three or more agents and a finite set of alternatives, then any surjective performance standard which satisfies both monotonicity and no veto power can be fully implemented in Nash equilibrium ("monotonicity" and "no veto power" will be defined in Sections 5 and 7, respectively). A hypothesis must be added to the statement of this theorem for Maskin's proof to be completely correct: the result as stated is true, but the proof in [7] does not enable one to conclude that the game which he constructs will fully implement the performance standard in question without the use of an additional hypothesis. In [15], it is shown that this extra hypothesis can in fact be omitted. The hypothesis that we will add to correct the error in Maskin's proof is a very "weak" hypothesis, however, so it is accurate to say that we will really only characterize an exception to Maskin's proof in this paper. This topic will be discussed in Section 9, where the relationship of Maskin's results to the results of this paper will be fully explained.

Finally, there are two more topics which should be mentioned in this sum-

mary. In Section 6, we will discuss how the necessary condition in Theorem 2 can be used to gain insight into the Nash implementation problem. Our goal in this section is to explain several ways that this condition can be interpreted, and also how it might provide valuable information to the designer of games.

All but one of the hypotheses of our construction process are easily verified in the case of the price mechanism. In the last section of this paper (Section 11), we will show that the price mechanism in fact satisfies all of the hypotheses of Theorem 7. We thereby show that our construction process applies to the price mechanism. This is not the real purpose of this section, however: our primary goal is to further explain one of our hypotheses, and also to show that it has some generality.

At this point, we can begin to answer the embedding problem which was first described at the beginning of this section.

4. Necessary Conditions for the Existence of an Embedding.

As above, let (μ, M, h) be a message process which (fully) realizes the performance standard F . Our first theorem provides a necessary condition that μ , M , and h must satisfy if there exists a game (τ, S, g) which contains (μ, M, h) . The proof of this theorem is trivial: it follows almost immediately from the definition of Nash equilibrium. The correspondences from M to A which the theorem defines, however, will motivate much of the analysis in the remainder of this paper.

Theorem 1. Let (μ, M, h) be a message process which (fully) realizes the performance standard F , and assume that there exists a game (τ, S, g) which contains (μ, M, h) . Then there exist correspondences $\gamma_1, \gamma_2, \dots, \gamma_n$ from M to A which have the following property:

for any $(e_i) \in E$ and $m \in M$,

$$m \in \mu((e_i)) \text{ if and only if } \gamma_j(m) \subset L(h(m), e_j) \text{ for each } 1 \leq j \leq n. \quad (6)$$

Proof. Given an element m of M , let (s_i^*) denote the element of S which it is identified with under the embedding of M into S . For $1 \leq j \leq n$, the correspondence γ_j is defined to be the set of all the alternatives which the j^{th} agent can attain by unilaterally changing his own strategy s_j away from s_j^* :

$$\gamma_j(m) = \{a \in A \mid a = g(s_j, s_{-j}^*) \text{ some } s_j \in S_j\}.$$

We can now verify (6). If m is an element of $\mu((e_i))$, then by (2) the profile of strategies $m = (s_i^*)$ is a Nash equilibrium. The following formula must therefore be true for each $1 \leq j \leq n$:

$$\gamma_j(m) \subset L(g(m), e_j) = L(h(m), e_j).$$

Conversely, if $\gamma_j(m)$ is a subset of $L(h(m), e_j)$ for each value of j , then m is a Nash equilibrium for the environment (e_i) . By (2), we can conclude that m must be an element of $\mu((e_i))$. Q.E.D.

Since the correspondences $\gamma_1, \dots, \gamma_n$ are necessary for the existence of a game form which contains the given message process, they must clearly play a role (at least implicitly) in the sufficient conditions for the existence of such a game which will be presented in Theorems 4-7. The reader should keep in mind that the value of γ_j at a particular element m of M is supposed to represent the set of all the alternatives which the j^{th} agent can achieve by unilaterally moving his "strategy" away from the value that is determined by m , even if the larger game (τ, S, g) which is supposed to contain (μ, M, h) has not yet been specified.

Even though the proof of Theorem 1 is hardly more than a straightforward application of the definition of a Nash equilibrium, the theorem is important, because it will enable us to determine if a given message process (μ, M, h) can be contained in some larger game. The weakness of this theorem is that the condition which it provides for checking to see if (μ, M, h) is "suitable" -- statement (6) -- depends on the existence or nonexistence of "unknown" correspondence $\gamma_1, \gamma_2, \dots, \gamma_n$ which have certain properties. At first glance, it may seem very difficult to determine whether or not these correspondences actually exist. Our next theorem will enable us to avoid this problem, however, for it eliminates the "unknowns" from (6) to produce a formula which can be checked directly with μ, M , and h .

Before stating this theorem, we must make several definitions. For any m in M , the subset $\mu^{-1}(m)$ of E is given by the following formula:

$$\mu^{-1}(m) = \{(e_i) \in E \mid m \in \mu((e_i))\}.$$

This defines a correspondence from M to E . Using μ , we now define correspondences $\xi_1, \xi_2, \dots, \xi_n$ from M to A in the following manner:

$$\text{for } m \in M \text{ and } 1 < j < n, \xi_j(m) \equiv \bigcap_{(e_i) \in \mu^{-1}(m)} L(h(m), e_j).$$

It is important to notice that each of the correspondences $\xi_1, \xi_2, \dots, \xi_n$ is completely defined in terms of μ, M , and h . It is also easy to show that any correspondences $\gamma_1, \gamma_2, \dots, \gamma_n$ which satisfy (6) must also satisfy $\gamma_j(m) \subset \xi_j(m)$ for every m in M and $1 < j < n$, since by (6) each $\gamma_j(m)$ must be contained in $L(h(m), e_j)$ whenever m is in $\mu((e_i))$.

Our next theorem will show that these "maximal" correspondences $\xi_1, \xi_2, \dots, \xi_n$ can be substituted for $\gamma_1, \gamma_2, \dots, \gamma_n$ in (6).

Theorem 2. Let (μ, M, h) be a message process which (fully) realizes the performance standard F . If there exists a game (τ, S, g) which contains (μ, M, h) , then the following formula must be true for any (e_i) in E and m in M :

$$m \in \mu((e_i)) \text{ if and only if } \xi_j(m) \subset L(h(m), e_j) \quad (7)$$

for each $1 < j < n$.

Proof. We begin by assuming that m is an element of $\mu((e_i))$. The conclusion that $\xi_j(m)$ must be a subset of $L(h(m), e_j)$ for each value of j follows immediately from the definitions of the correspondence $\xi_1, \xi_2, \dots, \xi_n$.

Conversely, suppose that (e_i) is an environment and m is a message such that $\xi_j(m)$ is a subset of $L(h(m), e_j)$ for each of the values of j . It was argued immediately before this proof that $\gamma_j(m)$ is a subset of $\xi_j(m)$ for each $1 < j < n$. Therefore, each of the sets $\gamma_j(m)$ must be contained in the corresponding set $L(h(m), e_j)$. We can now conclude that m is an element of $\mu((e_i))$ by applying (6). Q.E.D.

The implication "if $m \in \mu((e_i))$, then $\xi_j(m) \subset L(h(m), e_j)$ for each $1 < j < n$ " of (7) is a tautology, simply because of the way that we defined the sets $\xi_1(m), \xi_2(m), \dots, \xi_n(m)$. It is the converse of this statement which needs to be checked in order to determine if a game which contains (μ, M, h) can actually exist.

The reader should note that a "tension" exists between the implication "if $\xi_j(m) \subset L(h(m), e_j)$ for each $1 < j < n$, then $m \in \mu((e_i))$ " and the equation which guarantees that (μ, M, h) realizes F -- $h \circ \mu((e_i)) \in F((e_i))$. Formula (7) suggests that if we want to change (μ, M, h) so that it can serve as the set of Nash equilibria of some larger game, we only need to add some message values to $\mu((e_i))$ for certain values of the environment (e_i) . An example of this

will be given in Section 9. However, this cannot always be done, since a message value m cannot be added to $\mu((e_i))$ unless $h(m)$ is an element of $F((e_i))$. Roughly speaking, it may not be possible to "close off" the correspondence μ in order to make (μ, M, h) satisfy (7) without ruining this message process' ability to realize F .

5. The Relationship of Statement (7) to Monotonicity.

In this section, we will show that (7) implies that the performance standard F must be "monotonic" when (μ, M, h) fully realizes F .

First, we will define "monotonic". The performance standard F is monotonic when it satisfies the following property for all $(e_i), (e_i^*)$ in E and a in $F((e_i^*))$:

if for each $1 < j < n$ and any b in A the relation $a(>, e_j^*)b$ implies $a(>, e_j)b$, then a must be an element of $F((e_i))$. (8)

Maskin [8, p. 8] has expressed this property very succinctly in words: "... a (performance standard) is monotonic if, whenever an alternative a is in the choice set for a profile of preferences, and then those preferences are altered in a way such that a does not fall in anyone's preference ordering relative to any other alternative, it remains in the choice set."

In this same paper, Maskin shows that any performance standard which can be fully implemented in Nash equilibrium by some game must be monotonic. Since in our next section we will be discussing sufficient conditions on (μ, M, h) and F for the existence of a game which contains (μ, M, h) , it is worthwhile at this point to show how his necessary condition is related to our necessary condition. In fact, it is easy to show that the performance standard F must be monotonic if there exists a message process (μ, M, h) which satisfies (7) and fully realizes F . This will be the content of the following theorem.

Theorem 3. Let F be a performance standard and (μ, M, h) a message process which fully realizes F . If (μ, M, h) satisfies (7), the F is monotonic.

Proof. We assume that a is an element of $F((e_i^*))$, and that the statement " $a(>, e_j^*)b$ for each j and for any b in A " implies " $a(>, e_j)b$ for each j and for any b in A ". We must show that a is an element of $F((e_i))$. Using this implication, we can see that $L(a, e_j^*)$ is a subset of $L(a, e_j)$ for each value of j .

Since (μ, M, h) fully realizes F , there exists an element m^* of $\mu((e_i))$ which satisfies $h(m^*) = a$. By (7), the following is true for each value of j :

$$\xi_j(m^*) \subset L(h(m^*), e_j^*) .$$

Since $L(h(m^*), e_j^*)$ is a subset of $L(h(m^*), e_j)$ for each j , we can then apply (7) in the "reverse direction" to conclude that m^* must be an element of $\mu((e_i^*))$. The fact that (μ, M, h) fully realizes F then implies that a is in $F((e_i))$. Q.E.D.

We will return to this topic in Section 9 when we discuss the relationship of Maskin's construction to ours.

6. Further Discussion of Statement (7).

Before we move on to discuss our sufficient conditions, we will make several remarks about (7). These will truly be "remarks" and not "results", for they will not be proven statements: rather, they represent an attempt to gain insight into the problem of Nash implementation from (7).

First, note that (7) suggests how increasing the size of the messages space M can make it more likely that (μ, M, h) can be contained in some game which implements F in Nash equilibrium. Assume that (μ, M, h) does not satisfy (7):

for some m in M and (e_i) in E , $\xi_j(m)$ is a subset of $L(h(m), e_j)$ for all j , but m is not an element of $\mu((e_i))$. As we discussed earlier, it may not be possible to correct this problem by simply adding m to $\mu((e_i))$, since $h(m)$ might not lie in $F((e_i))$. The only solution is to change the "other" side of (7): (μ, M, h) must be redefined so that $\xi_j(m)$ is no longer a subset of $L(h(m), e_j)$ for all of the j values.

Recall our definition of $\xi_j(m)$:

$$\xi_j(m) \equiv \bigcap_{(e_i) \in \mu^{-1}(m)} L(h(m), e_j) .$$

We usually think of moving to a larger message space as a switch to a larger language: more "words" are added in such a way as to distinguish between environments which in the old language were indistinguishable. Mathematically, this means that when M is embedded in some larger message space the "size" of $\mu^{-1}(m)$ tends to be smaller, since $\mu^{-1}(m)$ consists of those environments which are identical from the viewpoint of our language. This means that as we move to a larger message space, the size of $\xi_j(m)$ should tend to increase. Making the language sufficiently large will insure (hopefully) that $\xi_j(m)$ will no longer be contained in $L(h(m), e_j^*)$ for at least one value of j . This of course does not constitute an algorithm, but it does suggest that a correlation exists between the size of the message space M and the likelihood that (μ, M, h) is contained in the set of Nash equilibria of some game which implements F .

Our next remarks concern the advantages which might accrue to the designer of games when he is allowed to use as large a set of alternatives as possible. There is commonly some choice in this matter in many economic settings: the agents' preferences are often defined on a larger set than just the set of those alternatives which are actually assigned to some environment. For instance, the preferences of each agent in a classical economy are defined not only over the set of "balanced" (or feasible) net trades, but also over the set of all

"unbalanced" (or unfeasible) net trades as well. A performance standard can only assign balanced net trades to an environment, since an unbalanced net trade is from a practical viewpoint either inefficient or unfeasible. Any performance standard which assigns only the Walrasian allocations will not even use all of the balanced net trades if the number of agents is strictly greater than the number of commodities.

Returning to the general case, if the set of alternatives A is strictly larger than $F(E)$, then it is possible for a game (τ, S, g) to implement F in Nash equilibrium even though the image of g is not contained in $F(E)$: it is only necessary that $g(NE(g))$ be a subset of $F(E)$. In some cases, it might even be necessary that g use alternatives which do not lie in $F(E)$.

A game which has this property may be regarded as undesirable, but this mostly depends upon how much confidence one has that the Nash solution concept is exactly the right one for the problem at hand. Our purpose here is neither to criticize nor defend the Nash solution concept; instead, we want to indicate mathematically how the freedom to use all of the alternatives in A could be very important to the designer of games.

This issue can be analyzed by using (7). Imagine that we have a message process (μ, M, h) which satisfies (7) when A is used as the set of alternatives, and that we now replace A with some proper subset B of A which contains $F(E)$. We want to consider only those games (τ, S, g) for which $g(S)$ is contained in B .

Let b be an element of B , and e_j an element of E_j . The crucial point of this analysis is that when we replace A with the smaller set B , the set $L(b, e_j)$ in (7) also changes: it may become smaller, since $L(b, e_j)$ must now be defined as the lower contour set of b in B rather than in A . In (7), the "old" $\xi(m)$ and $L(h(m), e_j)$ must be replaced with $\xi(m) \cap B$ and $L(h(m), e_j) \cap B$, respectively.

Using this fact and the elementary properties of sets, we can deduce the following:

if (μ, M, h) satisfy (7) when B is regarded as the set of alternatives, then (μ, M, h) satisfies (7) when A is used as the set of alternatives.

The converse of this statement is clearly false in the general case.

Strictly speaking, this only verifies something which is intuitively obvious -- that the "extra" freedom of using the larger set A can only be to the designer's advantage. It is important, however, because it suggests that (7) can be used as a mathematical tool for determining if the set of alternatives B is just too small for games which contain (μ, M, h) .

Statement (7) also suggests that there is a trade-off between the size of M and the size of A . If we start with B and find that (μ, M, h) does not satisfy (7) when it is expressed in terms of B , then we might still find that (μ, M, h) satisfies (7) when the larger set A of alternatives is used; consequently, we would not have to resort to increasing the size of M in the manner which was discussed earlier in this section. Again, this is what we would expect, but it is important to have a mathematical framework for discussing these issues.

Our last remarks concern both the preceding discussion and an assumption which will hold for the next four sections. Throughout our construction process, we will assume that (μ, M, h) satisfies (7) and that F is a surjective correspondence from E to A . At first glance, the assumption that $F(E) = A$ seems like an innocuous hypothesis: one could simply replace A with $F(E)$ when $F(E)$ happens to be a proper subset of A . This, in fact, is commonly done in the literature. Our preceding discussion, however, indicates that switching from A to $F(E)$ could jeopardize (7). The hypothesis that $F(E) = A$ is a strong requirement that we are imposing on F as a sufficient condition for our construction process, and it is reflected in another of our hypotheses for this construction -- statement (7).

7. Sufficient Conditions for the Existence of an Embedding

In this section, we will discuss the hypotheses about (μ, M, h) and F which we will need to construct a game that contains (μ, M, h) . Throughout our discussion of this construction (Sections 7-10), we will assume that (μ, M, h) fully realizes F , and that F is a surjective correspondence from E to A . We will also assume in the first part of this section that there are at least four agents -- $n \geq 4$.

Since our construction will be a generalization of Maskin's, our hypotheses should be related to his hypotheses. Theorem 3 has already established that (7) is in some sense a generalization of monotonicity, and Maskin also restricts his attention to the case where F maps E onto A . A more comprehensive discussion of how these two results are related to each other will be deferred until Section 9.

It may prove helpful if we briefly outline our construction process before we begin to discuss our hypotheses. We will define the set of strategies by letting $S_i = M$ for $1 \leq i \leq n$. M will be embedded into S by identifying it with the diagonal $D(M^n)$ in the usual manner, where $D(M^n)$ is the subset of M^n which can be characterized in the following ways:

$$\begin{aligned} D(M^n) &= \{(m_1, m_2, \dots, m_n) \mid m_j = m_k, 1 \leq j, k \leq n\} \\ &= \{(m, m, \dots, m) \mid m \in M\} . \end{aligned}$$

The challenge will be to extend the outcome function h to the points of M^n which do not lie on the diagonal in a way which will make the resulting game both contain (μ, M, h) and implement F in Nash equilibrium.

Our first hypothesis is a property of F -- no veto power. F satisfies no veto power if the following is true for all environments (e_i) and alternatives a :

if $a(>, e_j)b$ for any b in A and for all but at most one value of j , then a must be in $F((e_i))$. (9)

No veto power means that if for some environment (e_i) the alternative a is top-ranked among all elements in A by all but at most one of the agents, then this alternative must be an element of $F((e_i))$: the single agent who does not rank a as equal to or better than the other elements of A cannot veto it. A comprehensive explanation of why this property is typically satisfied by performance standards which arise in economic models can be found in either of Maskin's papers [7], [8], so we will not attempt to justify it further here.

We will next make an assumption about the message process (μ, M, h) :

for any a in A , $h^{-1}(a)$ contains at least two distinct message values. (10)

The importance of this hypothesis will not become clear until we actually prove Theorem 4. This is the simplest hypothesis that we can make in order to insure that our construction process will work successfully. In Section 10, we will discuss a hypothesis which can be substituted for this one. We have chosen to use (10) exclusively at this stage because it will enable us to avoid several complications in our construction process.

In practice, hypothesis (10) is true for many message processes. If $h^{-1}(a)$ consists of the single point m^* for some a in A , then $F^{-1}(a) = \mu^{-1}(m^*)$ must have the structure of a Cartesian product vis a' vis the sets of agent's characteristics:

if $\mu((e_i)) \equiv \bigcap_{i=1}^n \mu_i(e_i)$, then (11)

$$F^{-1}(a) = \mu_1^{-1}(m^*) \times \mu_2^{-1}(m^*) \times \dots \times \mu_n^{-1}(m^*) .$$

It is easy to show that in the case of C^2 performance standards, the primage of any point in the range of a generic F would not have the geometric structure which is required by (11) (see Williams [14]). Note, however, that the price mechanism (as formulated in Mount and Reiter [10]) will not satisfy (10) when the number of agents is greater than or equal to the number of commodities: in this case, a generic element of the set of all net trades which are supported by a price vector will in fact be supported by a unique (normalized) price vector. This is one of the reasons that we are interested in alternatives to hypothesis (10).

Our last assumption concerns the existence of a mapping. For the moment, we will discuss this assumption from a slightly more abstract viewpoint. Given an arbitrary set B , we will let B^n denote the n -fold Cartesian product of B . If $P_B: B^n \rightarrow B$ is some function and $(b_1, b_2, \dots, b_n)$ is an arbitrary element of B^n , we will let $P_B(\cdot, b_{-j}): B \rightarrow B$ denote the following mapping from B to B :

$$\text{for any } b \in B, P_B(b, b_{-j}) \equiv P_B(b_1, \dots, b_{j-1}, b, b_{j+1}, \dots, b_n) .$$

For our construction, we will need to assume that at least one of the following statements is true:

$$\begin{aligned} \text{i) there exists a mapping } P_M \text{ from } M^n \text{ to } M \text{ such that} \\ \text{for any } (m_1, m_2, \dots, m_n) \in M^n \text{ and } 1 < j < n, \text{ the mapping} \\ P_M(\cdot, m_{-j}) \text{ is surjective; or,} \end{aligned} \tag{12}$$

$$\begin{aligned} \text{ii) there exists a mapping } P_A \text{ from } A^n \text{ to } A \text{ such that} \\ \text{for any } (a_1, a_2, \dots, a_n) \in A^n \text{ and } 1 < j < n, \text{ the} \\ \text{mapping } P_A(\cdot, a_{-j}): A \rightarrow A \text{ is surjective.} \end{aligned} \tag{13}$$

Either one of these mappings will enable us to define a suitable game rule on M^n .

It will be convenient if we continue to discuss this hypothesis in terms of a set B and a mapping P_B from B^n to B , where B should now be understood to represent either M or A . As an example, note that a suitable function P_B will exist whenever B has (or can be given) the structure of a group [9, p. 143]. In this case, the value of P_B at $(b_1, b_2, \dots, b_n)$ can simply be defined as the value of the product $b_1 b_2 \dots b_n$ under the group multiplication. It is easy to verify that when P_B is defined in this manner, the mapping $P_B(\cdot, b_{-j}): B \rightarrow B$ will be surjective for any $(b_1, b_2, \dots, b_n)$ in B^n and for all values of j . This suggests that any finite set b of cardinality m could be handled by simply identifying it with the group of integers modulo m [9, p. 96], while any countable set B could be identified with the additive group of integers.

If B is a continuum, however, there will normally be some topological or even differential properties that are part of the model and which we will want P_B to possess. In these cases, it would not serve our purposes to impose some arbitrary group structure on B in order to define P_B . If B is equipped with a group structure which respects its geometric structure (i.e., if B is a Banach space, or a topological group), then this would not be an issue: the group structure of B could still be used to define P_B .

In the case of more general topological spaces B , however, other properties of B would have to be exploited in order to define a suitable mapping P_B . This will be illustrated by our discussion of the price mechanism in Section 9. It explains why we offer two alternative hypotheses, one for M and one for A : depending upon the structure of these two sets, it might be easier to find a suitable P_M than a suitable P_A (or vice-versa).

8. The Construction Process.

This section contains the proof that the hypotheses of Section 7 are sufficient to insure that a game can be constructed which contains (μ, M, h) . This proof will illustrate our construction process. The same process will be used again later in the proofs of several related results.

Theorem 4. Let F be a surjective performance standard from $\prod_{i=1}^n E_i$ to A (where $n > 4$), and let (μ, M, h) be a message process which fully realizes the performance standard F . Assume that F and (μ, M, h) have the following properties:

$$\text{i) } m \in \mu((e_i)) \text{ if and only if } \xi_j(m) \subset L(h(m), e_j) \text{ for each } 1 < j < n ; \quad (7)$$

$$\text{ii) } F \text{ satisfies no veto power;} \quad (9)$$

$$\text{iii) for any } a \in A, h^{-1}(a) \text{ contains at least two distinct message values;} \quad (10)$$

$$\text{iv) either } P_M \text{ or } P_A \text{ exists, and has the properties which are specified by (12) and (13) (respectively).} \quad (14)$$

Then there exists a game (τ, M^n, g) which contains (μ, M, h) .

Proof. Note that the strategy space of the game is specified in the statement of the theorem: we set $S_i \equiv M$ for each value of i . M is embedded into the strategy space M^n by means of the standard identification of M with $D(M^n)$:

$$\text{for any } m \in M, m \rightarrow (m, m, \dots, m) .$$

The outcome function h determines how the game form g must be defined on $D(M^n)$:

$$\text{i) for any } (m, m, \dots, m) \in D(M^n), g(m, m, \dots, m) \equiv h(m) . \quad (15)$$

Our problem is to define g at points of M^n which do not lie in $D(M^n)$. We will define g so that in addition to (14) it also has the following two properties:

for any (m_i^*) in M^n and any $1 < j < n$,

$$\text{ii) if } m_i^* = m_k^* \equiv m^* \text{ for all } i, k \neq j, \text{ then } g(\cdot, m_{-j}^*) \text{ maps } M \text{ onto } \xi_j(m^*) ; \quad (16)$$

$$\text{iii) if } m_q^* \neq m_k^* \text{ for at least two indices } q, k \neq j, \text{ then } g(\cdot, m_{-j}^*) \text{ maps } M \text{ onto } A . \quad (17)$$

Note that the image of the function $g(\cdot, m_{-j}^*)$ consists of all of the alternatives which the j^{th} agent can achieve by unilaterally changing his own strategy when the strategies of the other agents remain fixed at the values determined by (m_i^*) . Before we show how g can be defined so that (15), (16) and (17) are satisfied, we will show that these three conditions are sufficient to insure that (τ, M^n, g) contains (μ, M, h) .

We will first show that $\mu((e_i)) = \tau((e_i)) \cap M$. This will follow from (16). If m^* is in $\mu((e_i))$, then by (16) we can conclude that m^* must also be a Nash equilibrium for (e_i) , i.e., $m^* \in \tau((e_i))$. The converse is only slightly more difficult. If m^* is a Nash equilibrium for the environment (e_i) , then by the definition of a Nash equilibrium the following must be true for each value of j :

$$\{a \in A \mid a = g(m, m_{-j}^*), \text{ some } m \in M\} \subset L(a, e_j) . \quad (18)$$

By (16), however, the set on the left in (18) is equal to $\xi_j(m^*)$. Therefore, $\xi_j(m^*)$ is a subset of $L(a, e_j)$ for each value of j . By hypothesis (7), we can conclude that m^* is an element of $\mu((e_i))$.

We will now prove that (τ, M^n, g) implements F . This reduces to showing that if an element (m_i^*) of $M^n \setminus D(M^n)$ is a Nash equilibrium for the environment (e_i) , then $g((m_i^*))$ must be an element of $F((e_i))$. Since by assumption (m_i^*) is not an element of $D(M^n)$, by (17) any one of at least $(n-1)$ of the agents could achieve any of the other alternatives in A by unilaterally changing his own strategy. Consequently, by the definition of Nash equilibrium at least $n - 1$ of the agents must rank $g((m_i^*))$ as equal to or better than any alternative in A . Therefore, by no veto power, we can conclude that $g((m_i^*))$ must lie in $F((e_i))$.

Since (μ, M, h) fully realizes F , condition (14) above implies that (τ, M^n, g) must fully implement F . We have shown, then, that it is sufficient to find a g which equals h on $D(M^n)$ and also satisfies (16) and (17). We will now construct a game form g which has these properties.

Our definition of g will follow the order which is prescribed by (15), (16), and (17). Condition (15) defines g on $D(M^n)$, so we will now define g at all point of M^n which can be obtained by starting at a point in $D(M^n)$ and allowing a single agent to unilaterally alter his own strategy. For $m_1^* = m_2^* = \dots = m_n^* \equiv m^*$ and for each $1 < j < n$, we define $g(\cdot, m_{-j}^*)$ in the following manner:

$$\text{for any } m \in M, g(m, m_{-j}^*) = \begin{cases} h(m) & \text{if } h(m) \in \xi(m^*) \\ h(m^*) & \text{otherwise.} \end{cases} \quad (19)$$

This definition is consistent with (15) and it insures that (16) is satisfied.

Note that this definition is unambiguous as j varies only because $n \neq 2$: as long as $n > 2$, subsets of M^n which have the form

$$\{m_1^*\} \times \dots \times \{m_{j-1}^*\} \times M \times \{m_{j+1}^*\} \times \dots \times \{m_n^*\}$$

for arbitrary values of j and of (m_i^*) either do not intersect, intersect at a

unique point in $D(M^n)$, or else they are identical. For $n = 2$, however, distinct sets of this form can intersect at points which are not in $D(M^2)$. At these points, the mapping g might not be well-defined. This explains why this construction fails in the two agent case. Our hypothesis that $n > 4$ will be used momentarily.

We must now define g at all the remaining points in M^n -- those points (m_i^*) for which there exists at least three distinct elements of M in the set $\{m_1^*, m_2^*, \dots, m_n^*\}$. To define g over this subset of M^n , we will either use P_M or P_A , depending upon which of these two functions is presumed to exist. At any point (m_i^*) in this subset, the value of g is defined by whichever of the following two rules is appropriate:

$$\text{i) if } P_M \text{ is given, } g((m_i^*)) \equiv h \circ P_M((m_i^*)) . \quad (20)$$

$$\text{ii) if } P_A \text{ is given, } g((m_i^*)) = P_A(h(m_1^*), h(m_2^*), \dots, h(m_n^*)) . \quad (21)$$

This completes the definition of g .

To complete the proof of the theorem, we need to verify that g satisfies (17). In order to simplify this discussion, we will assume that (20) is used in defining g . The argument in the case where (21) is used is virtually identical to the one which we will make here, since the properties of the mappings P_M and P_A are so closely related to each other.

We need to show that if we start at an element (m_i^*) where $m_q^* \neq m_k^*$, then the j^{th} agent (where $q \neq j \neq k$) can achieve any of the alternatives in A by unilaterally varying his own strategy away from m_j^* . Two of our hypotheses -- $n > 4$, and $h^{-1}(a)$ contains at least two message values for any a in A -- will play a critical role in this argument. First, we assume that $h \circ P_M$ (rather than (19)) is the rule which is used for computing g at all the points in M^n that are reached as the j^{th} agent varies his strategy. By our assumption (12) about P_M , this mapping will map this set of strategies onto

M . Since h maps M onto A , we can then conclude that the j^{th} agent can achieve any alternative by varying his own strategy.

A difficulty arises when $h \circ P_M$ does not define g at all the points in M^n that are reached when the j^{th} agent varies his own strategy. For instance, if we begin at a point where $m_1^* \neq m_2^*$ but $m_r^* = m_2^*$ for all $r > 3$, then when the third agent adjusts his strategy, he will reach a point where (19) defines g when he sets his strategy equal to m_2^* . The argument in the preceding paragraph by itself would not enable us to conclude that this agent could achieve any alternative by changing his strategy.

If $m_q^* \neq m_k^*$ but $m_r^* = m_k^*$ for all $q \neq r \neq j$, then the rule $h \circ P_M$ will apply to all but one of the points in M^n that are reached when the j^{th} agent varies his strategy. For the moment, we will use $S^j(m_{-j}^*)$ to denote the elements of M^n that are "traced out" when the j^{th} agent varies his strategy:

$$S^j(m_{-j}^*) = \{(m, m_{-j}^*) \in M^n \mid m \in M\} .$$

The range of P_M over the subset of $S^j(m_{-j}^*)$ where the rule $h \circ P_M$ defines g might "miss" at most a single point of M . The range of $h \circ P_M$ over this set of strategies, however, will still equal A because of assumption (10) about h . We therefore can still conclude that the j^{th} agent can achieve every alternative in A by changing his own strategy. Q.E.D.

Our assumption that $n > 3$ plays a subtle role in the very last argument in this proof. When $n = 3$ and $m_q^* \neq m_k^*$, there would be exactly two strategies that the j^{th} agent could choose -- either m_q^* or m_k^* -- where (19) (rather than $h \circ P_M$) defines g . Consequently, the range of P_M over the subset of $S^j(m_{-j}^*)$ where $h \circ P_M$ defines g might miss as many as two of the elements of M . We therefore cannot conclude from (12) alone that the j^{th} agent can achieve any alternative in A by unilaterally changing his own strategy.

It might be possible to modify the definition of the game form g in a way which avoids this difficulty, but it is not clear how this can be accomplished in the general case. There is an obvious hypothesis which could be added to clear up this problem -- for any a in A , $h^{-1}(a)$ contains at least three messages values. This will be the content of the next theorem. Further discussion of how hypothesis (10) of Theorem 4 might be altered will be deferred to Section 10.

Theorem 5. Let F be a surjective performance standard from $\prod_{i=1}^3 E_i$ to A , and let (μ, M, h) be a message process which fully realizes F . Assume that (μ, M, h) and F satisfy hypothesis (7), (9), and (14) of Theorem 4, and additionally, that h has the following property:

iv)' for any $a \in A$, $h^{-1}(a)$ contains at least three distinct message values.

Then there exists a game (τ, M^3, g) which contains (μ, M, h) .

Proof. The proof of this theorem follows from the construction process in Theorem 4, and from the discussion which immediately precedes the statement of this theorem. Q.E.D.

There are two more problems which we can easily solve by closely examining our construction process. First, we wish to determine conditions under which a game (τ, S, g) can be found which matches a given message process (μ, M, h) . When this can be accomplished, we then want to know the following: can the game (τ, S, g) be defined in such a way that each of its Nash equilibria are "strict" Nash equilibria? An n -tuple of strategies (s_i^*) is a strict Nash equilibrium for the environment (e_i) if when any agent is allowed to freely change his own strategy, he can only achieve outcomes which he finds inferior to $g((s_i^*))$:

for all j ,

$$\{g(s_j, s_{-j}^*) \mid s_j \in S_j\} \subset \{a \in A \mid g((s_j^*))(>, e_j)a\} .$$

Our next theorem will describe sufficient conditions for each of these cases.

Theorem 6. Let F be a performance standard and (μ, M, h) a message process which fully realizes F . Assume that (μ, M, h) and F satisfy either the hypotheses of Theorem 4 or of Theorem 5, and assume that no veto power is a vacuous assumption for F : for any (e_i) in E , there is no alternative in A which is top-ranked by $n - 1$ of the agents. Then the game (τ, M^n, g) which is constructed to prove Theorems 4 and 5 matches (μ, M, h) . Each element of $NE(g)$ will be a strict Nash equilibrium if in addition to the above hypotheses, (μ, M, h) and F also satisfy the following condition:

for any $1 \leq j \leq n$ and $m \in M$,

$$\bigcap_{(e_i) \in \mu^{-1}(m)} \{a \in A \mid h(m)(>, e_j)a\} \neq \emptyset . \quad (23)$$

Proof. As mentioned above, we will prove this theorem simply by making several careful observations about our construction process.

First, note that no veto power is used in our construction process only once -- to show that if an element (m_i^*) of $M^n \setminus D(M^n)$ is a Nash equilibrium for some environment (e_i) , then $g((m_i^*))$ must be in $F((e_i))$. Our argument showed that $g((m_i^*))$ must be a top-ranked element of A for at least $n - 1$ of the agents, and hence (by no veto power) it must lie in $F((e_i))$. If no veto power is a vacuous assumption about F , then it is clear that whatever the environment, no element of $M^n \setminus D(M^n)$ could ever be a Nash equilibrium. Hence, $NE(g) = D(M^n)$ (= M under the identification), and (τ, M^n, g) therefore matches (μ, M, h) .

To make each element of $D(M^n)$ into a strict Nash equilibrium, we must change our definition of g at those points in M^n that can be reached by starting at a point in $D(M^n)$ and then changing a single agent's strategy. Let (m_i^*) be in $D(M^n)$, with $m^* = m_1^* = m_2^* = \dots = m_n^*$. For simplicity, let $\eta(m^*)$ denote the subset of $\xi_j(m^*)$ which is used in (23):

$$\eta(m^*) \equiv \bigcap_{(e_i) \in \mu^{-1}(m)} \{a \in A \mid h(m)(\cdot, e_j)a\}.$$

Choose an element $a^*(m^*)$ of $\eta(m^*)$. Before, we defined g at any point (m, m_{-j}^*) in the following manner:

$$g(m, m_{-j}^*) = \begin{cases} h(m) & \text{if } h(m) \in \xi(m^*) \\ h(m^*) & \text{otherwise.} \end{cases} \quad (19)$$

It is easy to verify that (m^*) becomes a strict Nash equilibrium for any environment (e_i) in $\mu^{-1}(m^*)$ if we modify (19) in the following way:

$$g(m, m_{-j}^*) = \begin{cases} h(m^*) & \text{if } m = m^* ; \\ h(m) & \text{if } m \in \eta(m^*) ; \\ a^*(m^*) & \text{otherwise.} \end{cases}$$

This modification will not disturb any of g 's other properties, so the construction will still be successful. Q.E.D.

9. Maskin's Construction.

In this section, we will discuss how our construction generalizes Maskin's [7] construction. In particular, we will want to compare the hypotheses of these two results.

First, we must digress a bit. Given any performance standard F on the set of environments E , there always exists a "trivial" message process which fully realizes F -- the direct revelation mechanism. The message set M of the

direct revelation mechanism is the set of all elements $((e_i), a)$ of $E \times A$ which satisfy $a \in F((e_i))$. The outcome function h is simply the restriction to M of the standard projection mapping of $E \times A$ onto A :

$$\text{for } ((e_i), a) \in M, h((e_i), a) \equiv a.$$

Finally, the message correspondence μ is given by the following formula:

$$\text{for any } (e_i^*) \text{ in } E, \mu((e_i^*)) = \{((e_i), a) \in M \mid (e_i) = (e_i^*)\}.$$

It is easy to verify that μ is a coordinate correspondence and that (μ, M, h) fully realizes F .

When the "amount" of information about an environment which is transferred to the message space is the criterion that is used to evaluate message processes, the direct revelation mechanism is the worst possible message process that can be used to fully realize F . It represents the complete transfer to each agent's characteristics to the message space. There is no efficient condensation or encoding of information about environments in this message process. Much of the research on the problem of mechanism design has in fact been devoted to trying to devise alternatives to this message process in the case of particular performance standards. The direct revelation mechanism does have a very important virtue, however: we always know how to construct it, and we are certain that it will fully realize F . For an arbitrary performance standard, the alternatives to this message process are usually very limited and can be very difficult to construct (see, for example, Williams [14]).

We can now begin to discuss Maskin's result from the viewpoint of this paper. As in this paper, he restricts his attention to monotonic, surjective performance standards F which satisfy no veto power. Though Maskin does not explicitly assume that the performance standard is surjective, it is clearly a necessary assumption, else his construction may not satisfy (17). Unlike this paper he assumes that the set of alternatives A is finite. Maskin suggests, however,

that the proof extends easily to cover the case where A is "infinite" (presumably, he means countable). With these assumptions about F and A , Maskin then constructs a game which he claims will fully implement F in Nash equilibrium.

Essentially, Maskin's construction is an application of the process which was outlined in the proof of Theorem 4 to the particular version of the direct revelation mechanism which fully realizes F . To be accurate, a modification of the direct revelation mechanism must be made before our procedure can be applied: the correspondence μ must first be "closed off", to prepare it for its role in a Nash implementation. This will be discussed below. Also, Maskin defines g at those profiles of strategies that are reached by starting at a point in $D(M^n)$ and allowing a single agent to vary his own strategy in a different manner than we do: he uses the properties of the integers modulo m in his definition, rather than doing what we did in formula (19). Each of these definitions is merely a different route to the same goal -- insuring that g satisfies (16). The differences in these definitions do not play a role in the remainder of either argument, and they will not have any importance in the remainder of this analysis.

The obvious question is why our paper seems to use hypotheses for its construction which Maskin [7] does not need. The answer to this question lies partly in the added generality of this paper, and partly in the fact that his proof overlooks several difficulties.

The hypotheses that we use in Theorems 4 and 5 and which Maskin does not mention are listed below in the order in which they will be discussed in this section:

$$\text{i) } m \in \mu((e_i)) \text{ if and only if } \xi_j(m) \subset L(h(m), e_j) \text{ for each } 1 < j < n ; \quad (7)$$

$$\text{ii) either } P_M \text{ or } P_A \text{ exists;} \quad (14)$$

iii) when $n > 4$, $h^{-1}(a)$ contains at least two message values (10)
for any $a \in A$;

iv) when $n = 3$, $h^{-1}(a)$ contains at least three message values (22)
for any $a \in A$.

We will show that properties (7) and (14) are implicitly satisfied in the case which Maskin investigates, while (10) and (22) characterize an exception to Maskin's result.

First, we will show that the correspondence μ of the direct revelation mechanism which Maskin uses to construct his game can be redefined in a manner which insures that (7) is satisfied: μ must first be "closed off" (in the sense described in Section 4) in order that it may play a role in a Nash implementation. Technically speaking, our construction applies to this redefinition of the direct revelation mechanism. For any (e_i^*) in E , we define $\bar{\mu}((e_i^*))$ in the following manner:

$$\bar{\mu}((e_i^*)) = \{((e_j), a) \in M \mid L(a, e_j) \leq L(a, e_j^*), \text{ all } 1 \leq j \leq n\}.$$

By using the n conditions " $L(a, e_j) \leq L(a, e_j^*)$ " separately, it is easy to show that $\bar{\mu}$ is a coordinate correspondence. It is also trivial to verify that this new message process $(\bar{\mu}, M, h)$ satisfies (7). Since F is monotonic, the message process $(\bar{\mu}, M, h)$ will still fully realize F (where h , the outcome function is unchanged). It is this redefined version of the direct revelation mechanism which Maskin's game actually contains.

Property (14) is also satisfied in the problem which Maskin analyzes because he assumes that A is either finite or countably infinite. His actual construction is carried out only when A is finite with cardinality equal to m . As mentioned in Section 5, A can in this case be given the (additive) group structure of the integers modulo m , and the mapping P_A can be defined in terms of the group operation. This is exactly what Maskin uses to define his

game rule. As he indicates, this method also extends to the case where A is countable. In this instance, the (additive) group structure of the integers can be assigned to the elements of A , and the group operation can again be used to complete the definition of the game rule. Maskin's proof does not easily extend to cover the case of an arbitrary set A , unless a mapping P_A which satisfies (13) happens to exist. We have made this into an explicit assumption in this paper. Our discussion of the price mechanism in Section 11 will indicate that this is a nontrivial hypothesis.

Last of all, we turn to (10) and (22). As in our proof of Theorem 3, Maskin's aim is to produce a game (τ, M^n, g) which has the following property for (m_i^*) in M^n :

$$\begin{aligned} &\text{if } m_q^* \neq m_k^* \text{ for at least two indices } q, k \neq j, \text{ then} \\ &g(\cdot, m_{-j}^*) \text{ maps } M \text{ onto } A. \end{aligned} \tag{17}$$

Hypotheses (10) and (22) of Theorems 4 and 5 (respectively) insure that (17) is satisfied by the game form which we construct. They are included to overcome a difficulty in the construction process. This problem is discussed near the end of the proof of Theorem 4 and also immediately before the statement of Theorem 5.

Maskin [7] simply does not take this difficulty into account. In order to insure that the game which he constructs will actually implement the performance standard in question, some hypothesis must be added to the statement of his theorem. This is shown by an example which can be found in [15]. The third section of that paper consists of an example wherein Maskin's procedure (as defined in [7, Thm. 5]) is applied to a performance standard which satisfies both monotonicity and no veto power, but not (24). It is then shown that the resulting game does not implement the performance standard in Nash equilibrium. This is not a counter-example to Maskin's theorem, nor does it prove that a different strategy space is needed to implement that particular performance

standard; rather, it indicates that there is a deficiency in his argument, a deficiency which we have chosen to correct in this paper by adding a hypothesis.

In particular, there are versions of (10) and (22) that are appropriate for the specific cases which Maskin studies:

$$\begin{aligned} &\text{if } n > 4 \text{ and } a \in A, \text{ then } F^{-1}(a) \text{ contains at least} \\ &\text{two distinct environments; if } n = 3 \text{ and } a \in A, \text{ then} \quad (24) \\ &F^{-1}(a) \text{ contains at least three distinct environments.} \end{aligned}$$

In our next section, we will discuss an alternative to (24) for Maskin's result which would work equally well to insure that his game will actually implement F .

Maskin has shown with an example [7, Ex. 2] that a performance can fail to satisfy no veto power and still be implementable in Nash equilibrium. This same example shows that (24) is not a necessary condition for F to be implementable. In order to illustrate how our construction process may fail when (24) is not satisfied, we will now attempt to apply it to the performance standard from Maskin's example. Our analysis will show that the construction process breaks down before it would even use no veto power because (24) does not hold.

It would be desirable to illustrate the difficulty that can arise in our construction process by applying it to a performance standard which satisfies both monotonicity and no veto power, but not (24). We have chosen to analyze this particular example here, however, because it is an especially simple illustration of the problem.

Example. Our set of alternatives A has two elements -- $A = \{a_0, a_1\}$. There are three agents, and each set E_j consists of two elements, one of which corresponds to " a_0 is strictly preferred to a_1 ", and the other to " a_1 is strictly preferred to a_0 ". For simplicity, we will write $E_j = \{>, <\}$, where ">" represents " $a_0 > a_1$ ", while "<" represents " $a_1 > a_0$ ".

The performance standard F assigns the alternative a_0 to every environment except one: when every agent prefers a_0 to a_1 , the value of the performance standard is a_1 . Note that this performance standard satisfies neither (24) nor no veto power.

When our construction process is applied to the direct revelation mechanism, a strategy for an agent consists of a list of each agent's preferences, together with the value of A which F assigns to that profile of preferences. For instance, $(\succ, \prec, \succ, a_0)$ represents a typical strategy for a single agent.

In order to define P_A , we identify a_0 with 0 and a_1 with 1. P_A is then defined to be addition modulo 2, i.e., $a_1 + a_0 = a_1$, $a_1 + a_1 = a_0$, $a_0 + a_0 = a_0$. Implicitly, this is what Maskin does in his proof.

We will now examine the value of g at several different profiles of strategies.

Agent 1	Agent 2	Agent 3	Value of g
$\succ, \succ, \succ, a_1$	$\succ, \succ, \succ, a_1$	$\succ, \succ, \succ, a_1$	a_1
$\succ, \succ, \succ, a_1$	$\prec, \succ, \succ, a_0$	$\succ, \succ, \succ, a_1$	a_1
$\succ, \succ, \succ, a_1$	$\prec, \succ, \succ, a_0$	any strategy other than $\succ, \succ, \succ, a_1$	a_1

Table 1. The value of g at several profiles of strategies.

In the first line of Table 1, each agent's strategy is given by $(\succ, \succ, \succ, a_1)$. By (15) the value of g at this profile must equal a_1 .

In the second line, the second agent has changed his strategy to $(\prec, \succ, \succ, a_0)$. By (16), the value of g must remain equal to a_1 (else the first profile would not be a Nash equilibrium for the environment $(\succ, \succ, \succ)$).

In the final line, the third agent varies his strategy away from $(\succ, \succ, \succ, a_1)$. Using the above definition of P_A , by (21) the value of g does not change from a_1 as the third agent changes his strategy. The formula in Maskin's proof computes exactly this same answer.

If we combine lines 2 and 3 of Table 1, we see that (17) is not satisfied by g : when the first and second agents' strategies are different, the third agent cannot necessarily achieve every alternative in A by varying the value of his strategy.

It may still be possible to argue that g fully implements F in Nash equilibrium; otherwise, g might be redefined in some way to insure that it implements F . The point, however, is that our (or Maskin's) construction process may fail when (24) is not satisfied.

10. An Alternative to Hypotheses (10) of Theorem 4.

Hypothesis (10) of Theorem 4 is needed only once in our construction process -- to show that g satisfies (17). In this section, we will discuss a hypothesis which can be used to replace (10) in Theorem 4, (22) in Theorem 5, and (24) in Maskin's result. The hypothesis will be discussed only in the context of Theorems 4 and 5, since Maskin's result has been shown to be a special case of these theorems.

We must first pinpoint the difficulty which (10) is intended to handle. Recall that in Theorem 4, we assume that $n > 4$. At any point (m_i^*) in M^n where $m_q^* \neq m_k^*$ and $q, k \neq j$, we want to insure (17) holds: the mapping $g(\cdot, m_{-j}^*)$ sends M onto A . At "most" points (m_i^*) in M^n where $m_q^* \neq m_k^*$, we can conclude that $g(\cdot, m_{-j}^*)$ maps M onto A by using the properties of P_M (or P_A) and the fact that (μ, M, h) fully realizes F . At certain points in M^n , however, this argument is invalid. When (m_i^*) satisfies $m_q^* \neq m_k^*$ and also satisfies $m_r^* = m_k^*$ for all $q \neq r \neq j$, we can only guarantee that the range of $g(\cdot, m_{-j}^*)$ contains all but at most one of the points in A if we do not assume (10) or some substitute for this hypothesis.

A more difficult problem arises in the $n = 3$ case. Here, for any (m_1^*, m_2^*, m_3^*) which satisfies $m_q^* \neq m_k^*$, we can only deduce from our definition of

g that the range of $g(\cdot, m_{-j}^*)$ contains all but at most two of the points in A.

We will not try to redefine g so that these difficulties will no longer occur. Instead, we will change (17) so that it reads as follows when $n > 4$ and when $n = 3$:

if $n > 4$ and $m_q^* \neq m_k^*$, then for all $q \neq j \neq k$ the range of $g(\cdot, m_{-j}^*)$ includes all but at most a single element of A ; (25)

if $n = 3$ and $m_q^* \neq m_k^*$ for at least two indices $q, k \neq j$, then the range of $g(\cdot, m_{-j}^*)$ includes all but at most two of the points in A . (26)

We have already shown that the game form which we construct will satisfy (25) when $n > 4$ and (26) when $n = 3$. We now presume (10) and (22) have been dropped from Theorem 4 and 5, respectively. Our goal now is to add a hypothesis to each of these theorems which will insure that (τ, M^n, g) still fully implements F in spite of the fact that g now satisfies (25) (or (26)) instead of (17).

Statement (17) is used in our construction to show that if (m_i^*) in $M^n \setminus D(M^n)$ is a Nash equilibrium for the environment (e_i) , then $g((m_i^*))$ is an element of $F((e_i))$. Our argument was that since (m_i^*) is not in $D(M^n)$, by (17) at least $n - 1$ of the agents could achieve any of the alternatives in A by unilaterally changing his own strategy. No veto power then enable us to conclude that $g((m_i^*))$ must be in $F((e_i))$.

For the moment, we restrict our attention to the $n > 4$ case. Without using (17), we can still conclude from (25) that the following is true at a point (m_i^*) in $M^n \setminus D(M^n)$ for all but at most one value of the index j :

the j^{th} agent can achieve all but at most one of the alternatives in A by unilaterally changing his own strategy.

The alternatives which the j^{th} agent cannot achieve might be his (unique) top-ranked alternative, so no veto power is no longer sufficient to enable us to conclude that $g((m_i^*))$ is in $F((e_i))$.

When $n = 3$, (26) implies that the following statement is true for all but at most one value of j when (m_1^*, m_2^*, m_3^*) is in $M^n \setminus D(M^n)$:

the j^{th} agent can achieve all but at most two of the alternatives in A by unilaterally changing his own strategy.

As in the $n > 4$ case, no veto power is no longer sufficient to enable us to reach the desired conclusion.

A hypothesis which will enable us to conclude that $g((m_i^*))$ must be in $F((e_i))$ is given below:

for any a in A and any e_j in E_j , the set $\{b \in A \mid b(>, e_j)a\}$ is either empty or contains at least three elements of A . (27)

Note that this hypothesis is satisfied when A is a topological manifold of positive dimension, and each agent's preference preordering is continuous (as defined in Debreu [1, p. 56]). Consequently, this hypothesis will be satisfied in many economic models.

When (27) is substituted for either (10) in Theorem 4 or (22) in Theorem 5, no veto power will still be sufficient to enable us to complete the proof that g implements F . This will be the content of the following theorem. Technically speaking, we only need to assume in (27) that the set $\{b \in A \mid b(>, e_j)a\}$ is either empty or contains at least two elements in order to handle the $n > 4$ case: the assumption that this set contains at least three elements is needed only when $n = 3$. To avoid redundancy, however, we will state this theorem only with the stronger hypothesis.

Theorem 7. Let F be surjective performance standard from $\prod_{i=1}^n E_i$ to A (where $n > 3$), and let (μ, M, h) be a message process which fully realizes F . Assume that (μ, M, h) and F satisfy hypotheses (7), (9), and (14) of Theorem 4, and that $\prod_{i=1}^n E_i$ and A satisfy (27) above. Then there exists a game which both contains (μ, M, h) and fully implements F in Nash equilibrium.

Proof. We will prove this theorem only in the $n > 4$ case; a similar argument can be made when $n = 3$.

We do not change the definition of the game (τ, M^n, g) . By the discussion which precedes this theorem, we only need to show that if an element (m_i^*) of $M^n \setminus D(M^n)$ is a Nash equilibrium for some environment (e_i) , then $g((m_i^*))$ must be in $F((e_i))$. This will be proven in the same manner as in the proof of Theorem 4: we will show that $g((m_i^*))$ is a top-ranked alternative for at least $n - 1$ of the n agents, and we will then apply no veto power to reach the desired conclusion.

As in our proof of Theorem 4, each of the $n-1$ agents j who satisfy (25) will be shown to have $g((m_i^*))$ as a top-ranked alternative. Here, however, (27) (rather than (17)) will be used to show that $g((m_i^*))$ is top-ranked by all but at most one of the agents.

Let agent j be any one of the $n-1$ agents who can achieve all but at most one of the alternatives in A by changing his own strategy. If agent j can achieve every alternative in A , then by our earlier arguments we can immediately conclude that $g((m_i^*))$ must be a top-ranked alternative for agent j . Otherwise, assume agent j cannot achieve the element a^* in A by changing his strategy.

Since $g((m_i^*))$ is a Nash equilibrium for the environment (e_i) , we know that agent j ranks $g((m_i^*))$ as equal to or better than every element in A except perhaps a^* . Assume that agent j prefers a^* to $g((m_i^*))$. By (27),

there exists an element $a' \neq a^*$ in A such that agent j prefers a' to $g((m_i^*))$. This contradicts (25) and our assumption that (m_i^*) is a Nash equilibrium for (e_i^*) . Therefore, $g((m_i))$ must be a top-ranked alternative for agent j . Q.E.D.

11. An Example of Defining P_M .

Our purpose in this section is to illustrate how a mapping P_M which satisfies (12) can be constructed when neither M nor A has an obvious group structure. The price mechanism is the example which we will analyze.

The existence of either P_M or P_A is the only hypothesis of Theorem 8 which is difficult to verify in the case of the price mechanism (if the standard assumptions are made about the space of environments). When we define a suitable P_M for the price mechanism below, we will thereby show the existence of a game with finite-dimensional strategy space which fully implements the Walrasian allocations in Nash equilibrium. This is not really the point, however, since other research seems to indicate that a strategy space which has a much smaller dimension than the one that our construction creates can be used to fully implement the Walrasian allocations in Nash equilibrium. Our goal is to give insight into how to find suitable mappings P_M and P_A , and also to suggest thereby that our construction process has some generality.

First, we will describe those aspects of the price mechanism which will concern us in this section. Our discussion here centers on M and A alone, so we will neither discuss the space of environments, the definition of the Walrasian choice rule, nor the agents' message correspondences. These topics have been amply discussed elsewhere: a comprehensive discussion of the price mechanism for Cobb-Douglas environments can be found in Mount and Reiter [10], for instance.

We assume that there are n agents and ℓ commodities. The space P of normalized prices is given below:

$$P = \{p \in R^{\ell+} \mid \|p\| = 1\} .$$

Here, $\|\cdot\|$ denotes the standard Euclidean norm. We will use x_i to denote an element of R^{ℓ} , and (x_i) an element of $R^{n\ell}$. The message space M for the price mechanism is the subset of $P \times R^{n\ell}$ which consists of all pairs $(p, (x_i))$ that have the following properties:

$$i) \quad p \cdot x_j = 0, \text{ all } 1 \leq j \leq n; \quad (28)$$

$$ii) \quad \sum_{i=1}^n x_i = 0. \quad (29)$$

Roughly speaking, for a given p in S the ℓ -vector x_j corresponds to the j^{th} agent's excess demand when prices are determined by p . Property (28) is therefore the j^{th} agent's budget constraint, and (29) insures that in equilibrium a proposed trade is feasible and efficient.

Our set of alternatives A is the subset of $R^{n\ell}$ which consists of all balanced net trades that can be sustained by a price vector: (x_i) is an element of A if and only if

$$i) \quad \sum_{i=1}^n x_i = 0 \quad ((x_i) \text{ is balanced});$$

$$ii) \quad \text{there exists } p \text{ in } P \text{ such that } p \cdot x_j = 0 \text{ for all } 1 \leq j \leq n .$$

The outcome function h for the price mechanism is the restriction to M of the standard projection from $P \times R^{n\ell}$ to $R^{n\ell}$:

$$\text{for any } (p, (x_i)) \text{ in } M, h(p, (x_i)) \equiv (x_i) .$$

There is no obvious group structure on either M or A . We can still, however, define a mapping P_M which satisfies (12). Recall that P_M is supposed to send M^n onto M and also have the following property:

for any $(m_1, m_2, \dots, m_n) \in M^n$, the mapping

$g(\cdot, m_{-j}): M \rightarrow M$ is surjective.

We first must define a mapping ζ from P^n to P which will be used to define P_M . For $(p^j) \equiv (p^1, p^2, \dots, p^n)$ in S^n , $\zeta((p^j))$ is given by the following formula:

$$\zeta((p^j)) \equiv \frac{\sum_{j=1}^n p^j}{\|\sum_{j=1}^n p^j\|}.$$

Second, for v_1, v_2 in R^ℓ , we define $pr(v_1, v_2)$ to be the ℓ -vector obtained by subtracting from v_1 its component in the direction of v_2 . Obviously, $v_2 \cdot pr(v_1, v_2) = 0$.

Note that if (x_i) satisfies $\sum_{i=1}^n x_i = 0$, then $\sum_{i=1}^n pr(x_i, p) = 0$ for any p in P . This can be shown by a straightforward computation.

We can now define the mapping P_M . We will use $((p^j, (x_i^j)))$ to denote the following element of M^n :

$$((p^j, (x_i^j))) \equiv ((p^1, (x_i^1)), (p^2, (x_i^2)), \dots, (p^n, (x_i^n))).$$

Here, $(p^j, (x_i^j))$ represents the j^{th} agent's strategy. The value of P_M at $((p^j, (x_i^j)))$ consists of a normalized price vector p and a balanced set of net trade vectors (x_i) , each of which satisfies $p \cdot x_i = 0$. The price vector p in $P_M((p^j, (x_i^j)))$ is given by ζ :

$$p \equiv \zeta((p^j)).$$

The element (x_i) of $R^{n\ell}$ is given by the following formula:

$$(x_i) \equiv (\sum_{j=1}^n pr(x_i^j, p)).$$

Since $p \cdot \text{pr}(x_i^j, p) = 0$, $p \cdot x_i = 0$ for each value of i . A direct computation also proves that (x_i) is balanced. Therefore, $P_M((p^j, (x_i^j)))$ is in M . Finally, it can also be shown by a computation that P_M will satisfy (12).

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