POINCARÉ'S PROOF OF POINCARÉ'S LAST GEOMETRIC THEOREM

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POINCARÉ'S PROOF OF POINCARÉ'S LAST GEOMETRIC THEOREM

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Introduction. In this note we present a proof of the fixed point theorem known as Poincaré's Last Geometric Theorem. This theorem states that area preserving diffeomorphisms of the annulus which rotate the outer boundary clockwise and the inner boundary counter-clockwise must have a fixed point. It continues to play an important role in the study of Hamiltonian systems, differential equations and diffeomorphisms in spaces of low dimension (see [F2]).

Since Poincaré conjectured this theorem in his paper "Sur un théorème de géométrie" in 1912 [P], there have been many different proofs from the first, given by Birkhoff in 1913 [B1] to recent work of John Franks [F1]. Franks' version of the theorem greatly reduces both the geometric "twist" and the analytic area preservation hypotheses to obtain a theorem much stronger than Poincaré's original conjecture.

The purpose of this note is to review the original work of Poincaré in his 1912 paper. This contains not only the conjecture, but the proof of Poincaré's theorem in some special cases. The ideas used by Birkhoff and subsequent authors are considerably different from those of Poincaré (except for the attempt of Dantzig, see Birkhoff [B2]). In this note we show that Poincaré's original ideas can be modified slightly to yield a proof of his conjecture and give some interesting insights into the dynamics of these maps. The version of the theorem which we prove here is much weaker than Franks' theorem. We emphasize that this is **not** a correction of Poincaré's work – Poincaré was very clear on what he had and had not proven. This is rather an exposition and slight extension of Poincaré's ideas.

The authors would like to thank all those who listened patiently and assisted generously in this work, particularly Danny Goroff. The second author would also like to thank the Mathematics Department of the University of Cincinnati for its hospitality.

Definitions, Notations and Statement of the Theorem. We let

$$\begin{aligned}
\mathcal{A} &= S^1 \times [0,1], \\
A &= \mathbb{R} \times [0,1], \\
\pi &: A \to \mathcal{A} : (x,y) \to (2\pi(x \mod 1), y),
\end{aligned}$$

$$\left.\begin{matrix}\pi_x \\ \pi_y\end{matrix}\right\} : A \to \mathbb{R} : (x,y) \to \left\{\begin{matrix} x \\ y \end{matrix}\right. .$$

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denote the annulus, its universal cover the strip and the natural projection maps, respectively.

Notation : For $\tilde{f} : \mathcal{A} \to \mathcal{A}$ a diffeomorphism, we denote a (choice of) lift for $\tilde{f}$ by $f : A \to A$, so $\forall (x, y) \in A, f(x+1, y) = f(x, y) + (1, 0)$.

DEFINITION. A diffeomorphism $\tilde{f} : \mathcal{A} \to \mathcal{A}$, or its lift $f : A \to A$ will be called a **twist map** if

(1) $\tilde{f}$ is isotopic to the identity,

(2) $\forall x \in \mathbb{R}, \pi_x(f(x,0)) < x$ and $\pi_x(f(x,1)) > x$,

(3) $\tilde{f}$ preserves an absolutely continuous finite invariant measure with support all of $\mathcal{A}$.

Remarks. 1) We have included condition (3) in the above definition to shorten the phrase "area preserving twist map" to "twist map". Also condition (2) is sometimes called the "boundary twist condition".

2) There may be several choices of the lift f which satisfy condition (2) (as well as infinitely many choices for which it is not satisfied). We assume that both $\tilde{f}$ and f are specified.

3) The smoothness of f is not important – the theorems below hold with f a homeomorphism. (e.g. see Franks [F2])

4) We could also work on the infinite cylinder (or its cover $\mathbb{R}^2$) but the boundary curves simplify exposition greatly.

Poincaré's Last Geometric Theorem. Suppose $\tilde{f} : \mathcal{A} \to \mathcal{A}$ with lift $f : A \to A$ is a twist map. Then f has a fixed point.

Remarks. This is the theorem of Birkhoff (1913 [B1]). Birkhoff later improved this theorem, showing that such an $\tilde{f}$ has at least two fixed points ([B2], see also [BN]) and recent work of Franks [F2] have greatly weakened all of these hypotheses (see above).

Poincaré's idea in [P] was to study the set of points in A whose y-coordinate was not changed by the given twist map f. His goal was to show that if f has no fixed points then this set could be used to construct a loop in $\mathcal{A}$ which was contained in the exterior or interior of its image under $\tilde{f}$. Such a loop contradicts the area preservation hypothesis and hence every twist map must have a fixed point. Hence we need to develop some machinery for constructing the required curve. The following section developes the topological machinery. Next the rules for the (inductive) construction of the required loop are described.

Topological Preliminaries and Lemmas. We will use the following:

DEFINITION. If $\gamma : S^1 \to \mathbb{R}^2$ is a continuous map, for each point $z \in \mathbb{R}^2 \sim \gamma(S^1)$ we define the index of z to be the degree of the map $S^1 \to S^1 : t \to \frac{\gamma(t)-z}{||\gamma(t)-z||}$ where $||\cdot||$ is the usual $\mathbb{R}^2$ norm.

Remark. This is the usual winding number of a loop in $\mathbb{R}^2$ (hence a continuous, integer valued function on $\mathbb{R}^2 \sim \gamma(S^1)$). For definiteness we choose orientations so that a counterclockwise circle has index +1 about points in its interior.

DEFINITION. Let $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^2$ be two continuous arcs and let $\gamma_1 - \gamma_2$ denote the map $\gamma_1 - \gamma_2 : [0,4] \to \mathbb{R}^2$ given by

$$t \to \begin{cases} \gamma_1(t) & \text{if } 0 \le t \le 1 \\ (2-t)\gamma_1(1) + (t-1)\gamma_2(1) & \text{if } 1 \le t \le 2 \\ \gamma_2(3-t) & \text{if } 2 \le t \le 3 \\ (4-t)\gamma_2(0) + (t-3)\gamma_1(0) & \text{if } 3 \le t \le 4 \end{cases}.$$

We say $\gamma_1 - \gamma_2$ has *positive index* if it has positive or zero index about every point of $\mathbb{R}^2 \sim (\gamma_1 - \gamma_2)([0,4])$. (See Figure 1).

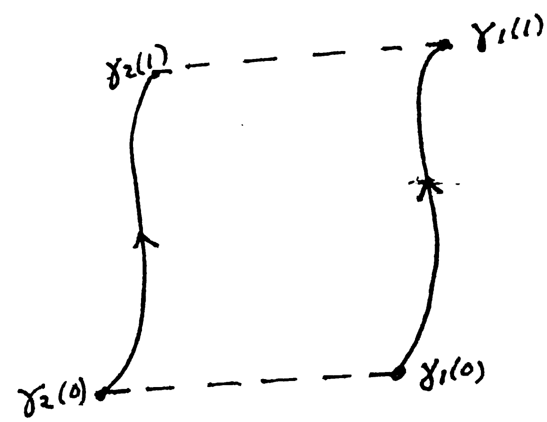

Figure 1. $\gamma_1 - \gamma_2$ has positive index.

Notation : For $\gamma, \delta : S^1 \to \mathbb{R}^2$ simple closed loops, we say $\gamma - \delta$ *has positive index* if when we consider $\gamma, \delta : [0, 2\pi] \to \mathbb{R}^2$ as arcs then $\gamma - \delta$ has positive index in the sense above. (This involves a rescaling of the domain of $[0, 2\pi]$ to $[0,1]$ in the obvious way).

LEMMA 1. *Suppose $\gamma, \delta : S^1 \to \mathbb{R}^2$ are simple closed loops. If there exist arcs $\gamma_i, \delta_i : [0,1] \to \mathbb{R}^2, i = 1, \ldots, n$ satisfying*

(1) $\forall i = 1, \ldots, n-1, \gamma_i(1) = \gamma_{i+1}(0), \delta_i(1) = \delta_{i+1}(0)$ *and* $\gamma_n(1) = \gamma_1(0), \delta_n(1) = \delta_1(0)$,

(2) $\cup_{i=1}^{n}\gamma_i([0,1]) = \gamma(S^1), \cup_{i=1}^{n}\delta_i([0,1]) = \delta(S^1)$,

(3) $\gamma_i - \delta_i$ *is positive index for* $i = 1, \ldots, n$,

then $\gamma - \delta$ *has positive index.*

Proof. This follows easily from the additivity properties of index. □

LEMMA 2. *Suppose* $\gamma, \delta : S^1 \to \mathbb{R}^2$ *are simple closed loops with* γ *and* δ *both having index* $+1$ *or zero about every point of* $\mathbb{R}^2 \sim \gamma(S^1)$ *or* $\mathbb{R}^2 \sim \delta(S^1)$, *respectively. Let* $U = \{z \in \mathbb{R}^2 \sim \gamma(S^1)$: *the index about* z *is* $+1\}$. *If* $\gamma - \delta$ *is positive index than* $\delta(S^1) \subseteq$ *closure* (U).

Proof. Since this index of points with respect to $\gamma - \delta$ is the difference of their indices with respect to γ and with respect to δ, each point of positive index of δ must also be a point of positive index of γ. Since δ is a simple closed curve, this implies $\delta(S^1) \subseteq$ closure (U). □

Poincaré's Idea for the Proof of Poincaré's Last Geometric Theorem. Fix $\tilde{f} : \mathcal{A} \to \mathcal{A}$ a twist map with lift $f : A \to A$. Poincaré's idea was to assume that f has no fixed points, then construct a curve $L \subseteq A$ which is either a simple closed loop, or has $\pi(L)$ a simple closed loop in $\mathcal{A}$, such that $L - f(L)($ or $\pi(L) - \tilde{f}(\pi(L)))$ has positive index. This contradicts the assumption of area preservation for f because lemma (3) implies that the region "inside" $L($ or $\pi(L))$ will be mapped inside itself by f (or $\tilde{f}$). (see Figure 2).

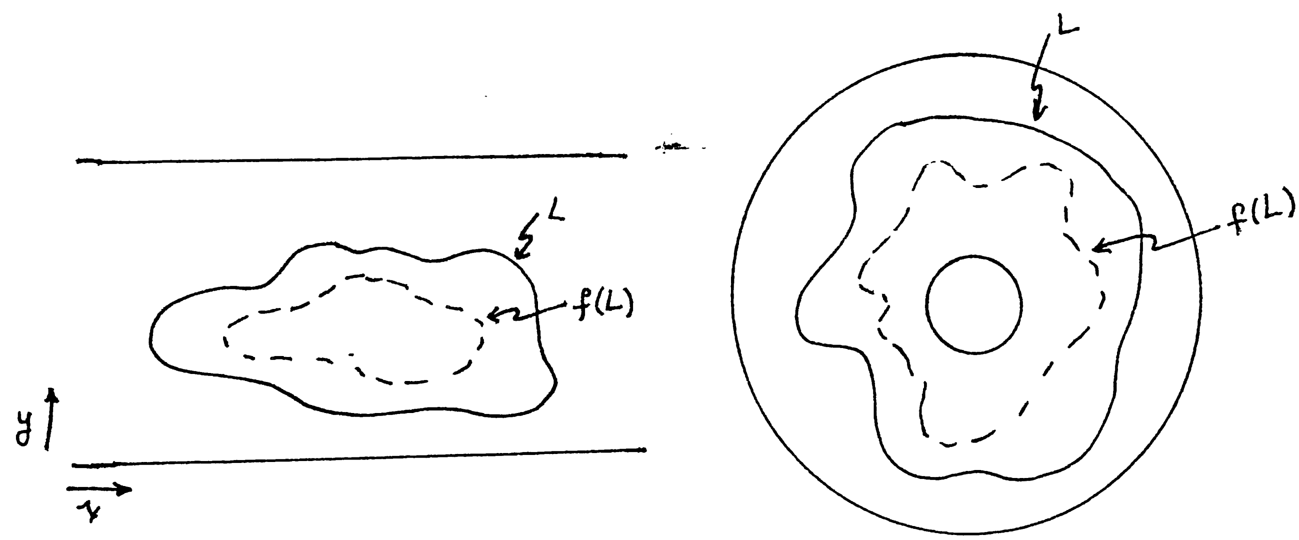

Figure 2.

To describe Poincaré's construction we need to define

$$\Gamma_f = \Gamma = \{z \in A : \pi_y(f(z)) = \pi_y(z)\}.$$

Typically, Γ will be a nice set. In fact, we have

LEMMA 3. *The twist maps $\tilde{f} : A \to A$ satisfying the following conditions are dense in the C^2 topology.*

(1) $\forall \varepsilon > 0, \Gamma_f \cap \{(x,y) \in A : \varepsilon < y < 1-\varepsilon, 0 \leq x \leq 1\}$ *is an immersed one manifold with finitely many components.*

(2) $\forall \varepsilon > 0, \Gamma_f \cap \{(x,y) \in A : \varepsilon < y < 1-\varepsilon, 0 \leq x \leq 1\}$ *has finitely many points with tangent parallel to* $(0,1)$, *all with different* y *coordinate,*

(3) $\forall \varepsilon > 0$, *the sign of* $\pi_y(f(z)) - \pi_y(z)$ *is different between any two adjacent components of* $\{(x,y) \in A : \varepsilon < y < 1-\varepsilon\} \sim \Gamma_f$.

Proof of lemma 3. We recall a few facts from transversality theory. For more details, we refer to [DeM, P], chapter 1, of which the following is taken (replacing C^∞ by C^1).

Let Λ, M, N be manifolds and let $F : \Lambda \times M \to N$ be a C^1 map. For $\lambda \in \Lambda$, we denote by $F_\lambda : M \to N$ the map defined by $F_\lambda(p) = F(\lambda, p)$. Let $S \subset N$ be a C^1 submanifold and let $T_S \subset \Lambda$ be the set of points λ such that F_λ is transversal to S.

PROPOSITION (3.3 P.25 IN [DEM, P]). *If $F : \Lambda \times M \to N$ is transverse to $S \subset N$, then T_S is residual in Λ.*

We can replace C^∞ by C^1, by using the C^1 version of Sard's theorem which is the cornerstone of the proof of the proposition. To apply this proposition in our situation, we will consider the map:

$$F : \mathbb{R}^2 \times A_\epsilon \to \mathbb{R} \quad \text{defined by}$$
$$F(\lambda, \mu, x, y) = \pi_y(g_{\lambda\mu}(x,y)) - y$$

where $A_\epsilon = \{(x,y) \in A \quad | \quad \epsilon < y < 1-\epsilon, 0 \leq x \leq 1\} and g_{\lambda\mu}$ is the monotone twist map generated by the function $H_{\lambda\mu}(x, \bar{x}) = \frac{1}{2\epsilon}(\bar{x}-x)^2 + \frac{\lambda}{2\pi}\cos 2\pi x + \frac{\mu}{2\pi}\sin 2\pi x$ where λ and μ are $0(\epsilon^2)$, $H_{\lambda\mu}$ can be extended to all of A in such a way as to generate a monotone twist map $g_{\lambda,\mu}$ satisfying:

$$g_{\lambda\mu}|_{A_\epsilon}(x,y) = (x + \epsilon(y - \lambda\sin 2\pi x + \mu\cos 2\pi x), y - \lambda\sin 2\pi x + \mu\cos 2\pi x)$$
$$g_{\lambda\mu}|_{A^c_{\epsilon/2}}(x,y) = (x + \epsilon y, y)$$

hence $g_{\lambda\mu} \circ f$ is a twist map and $\|g_{\lambda\mu} \circ f - f\|_{c^2} \sim 0(\epsilon)$ (One continues the constant λ, μ as functions of $(\bar{x} - x)$ satisfying $\|\lambda\|_{C^2}$, $\|\mu\|_{C^2} \sim 0(\epsilon)$ and with graphs of the form:

Denote by $f_1(z) = \pi_x f(z), f_2(z) = \pi_y f(z)$. Then, on $\mathbb{R}^2 \times A_\epsilon$, we have:

$$F(\lambda, \mu, z) =$$
$$f_2(z) - \lambda\sin(2\pi f_1(z)) + \mu\cos(2\pi f_2(z)) - \pi_y(z)$$

We need to show that, for a generic set of λ, μ, F is transverse to the 0-dimensional manifold $\{0\}$, i.e., according to the proposition that DF is always onto $\mathbb{R}$.

But

$$\frac{\partial F}{\partial \lambda}(\lambda, \mu, z) = \quad \sin(2\pi f_1(z))$$
$$\frac{\partial F}{\partial \mu}(\lambda, \mu, z) = \quad \cos(2\pi f_1(z))$$

which can't be simultaneously 0.

Hence, for generic values of (λ, μ), $F^{-1}_{(\lambda,\mu)}(0)$ is a 1-dimensional manifold. The proofs of the other two statements are similar. For the second statement, one consider the 0 set of the map $A \to \mathbb{R}^2$ given by:

$$z \to (f_2(z) - \pi_y(z), f_2'(z)).$$

One needs 4 independent parameters to unfold the singularities. Take:

$$h_{\lambda\mu\bar{\lambda}\bar{\mu}} = \frac{1}{2\epsilon}(\bar{x} - x)^2 + \frac{\lambda}{2\bar{n}} \cos 2\pi x + \frac{\mu}{2\bar{u}} \sin 2\pi x + \frac{\bar{\lambda}}{4\bar{u}} \cos 4\pi x + \frac{\bar{\mu}}{4\bar{u}} \sin 4\pi x$$

we leave the details to the reader. □

Also, we can easily see that the structure of Γ is related to the fixed points of f by the following:

LEMMA 4. *Suppose $\tilde{f} : A \to A$ is a twist map. Then there exists an $\varepsilon > 0$ such that if Γ contains a (connected) component $\Gamma_0 \subseteq \Gamma_f$ which connects the line $y = \varepsilon$ to the line $y = 1 - \varepsilon$ then Γ_0 contains a fixed point of $\tilde{f}$.*

Proof. When $\varepsilon > 0$ is sufficiently small we have $\forall x \in \mathbb{R}, \pi_x(f(x, \varepsilon)) < x$ and $\pi_x(f(x, 1-\varepsilon)) > x$. But, if Γ_0 connects $y = \varepsilon$ to $y = 1 - \varepsilon$ then for some $z_0, z_1 \in \Gamma_0$, we have $\pi_x(f(z_0)) < \pi_x(z_0)$ and $\pi_x(f(z_1)) > \pi_x(z_1)$. So, by continuity, there exists $z \in \Gamma_0$ with $\pi_x(f(z)) = \pi_x(z)$. But then z is the required fixed point because $z \in \Gamma_0$ implies $\pi_y(f(z)) = \pi_y(z)$, so $\tilde{f}(z) = z$. □

Now, the loop L above will be constructed from segments of Γ and horizontal ($y =$ constant) line segments contained in lines tangent to Γ such that each segment $\ell \subseteq L$ has $\ell - f(\ell)$ positive index. Poincaré's method was to construct a large, oriented graph in A from segments of Γ and horizontal segments in lines tangent to Γ with edges oriented so that an edge minus its image under f has positive index. If this graph contains an oriented loop then this is the required loop L. Unfortunately, Poincaré could not show that his directed graph contained an oriented loop (although he worked out numerous examples which yielded the required loop).

He states that if there is no such loop, he could construct a counterexample to the theorem. (Hence, he showed the existence of the loop in his graph is equivalent to the fixed point theorem).

We alter the construction slightly, building the curve L by adding arcs ℓ_i inductively to a curve in A such that $\ell_i - f(\ell_i)$ has positive index and ℓ_i is either an arc of Γ or a horizontal segment on a line tangent to Γ with end points in Γ. When this curve closes (as it must by the generic finiteness assumption on Γ) in A or $\mathcal{A}$, the resulting curve will be the required loop. This is really just a slight modification of the last step of Poincaré's construction.

We note that if we can prove the theorem for maps satisfying the conditions of lemma 3, then the theorem for arbitrary twist maps follows by the usual limit arguments (i.e., the limit of maps on a compact space having fixed points will have a fixed point).

Construction of L. Fix a twist map $\tilde{f} : \mathcal{A} \to \mathcal{A}$ with lift $f : A \to A$ which satisfies the conditions in Lemma (3) (i.e., Γ is made of smooth arcs etc.). We assume, for contradiction, that f has no fixed points. We will construct the loop $L \in A$ discussed above by piecing together inductively arcs ℓ_i such that $\ell_i - f(\ell_i)$ are positive index. The arcs ℓ_i will be pieces of Γ or horizontal segments on lines tangent to Γ, so eventually this curve will form a loop in A or its projection will form a loop in $\mathcal{A}$. The steps in this construction are the following:

I) Orient and label the components of Γ and $A \sim \Gamma$,

II) Describe the rules for choosing ℓ_{n+1} given $\ell_0, \ldots, \ell_n$,

III) Prove that ℓ_{n+1} exists and $\ell_{n+1} - f(\ell_{n+1})$ has positive index,

IV) Show the construction yields the required loop.

Step I: First we label the components of $A \sim \Gamma$ either "up" , if $\pi_y(f(z)) > \pi_y(z)$ or "down", if $\pi_y(f(z)) < \pi_y(z)$ for z in the component.

Next we label components of Γ as either "left" if $\pi_x(f(z)) < \pi_x(z)$ or "right" if $\pi_x(f(z)) > \pi_x(z)$ for z in the component. To orient components of Γ we first attach to each point $z \in \Gamma$ a vector $\underline{n}_z$ normal to Γ such that if z is on a left component of Γ then $\underline{n}_z$ points into an up region of $A \sim \Gamma$ while if z is on a right component of Γ then $\underline{n}_z$ points into a down component of $A \sim \Gamma$. Now we orient components of Γ by choosing a vector tangent to Γ at $z \in \Gamma$, called $\underline{t}_z$, such that $(\underline{t}_z, \underline{n}_z)$ has the same orientation as the standard basis $((1,0),(0,1))$. (See Figure 3).

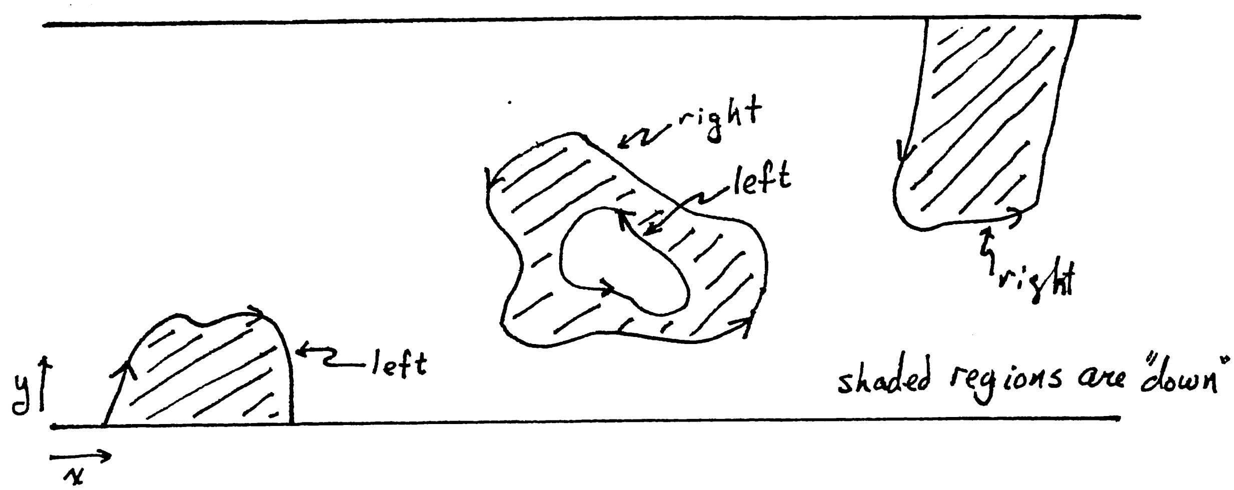

Figure 3.

Next we must break Γ into two disjoint subsets – the points of Γ whose tangent vectors are rotated by f a large amount and those for which the tangent vectors are rotated only a small amount. The fact that f preserves the y coordinate of Γ will allow us to distinguish "large" from "small" amounts of rotation as follows: To each non-zero tangent vector on A assign an angle given by the absolute value of angle between the vector and its image under the derivative of f. In order to remove the ambiguity mod 2π in this choice we require that the tangent vectors tangent to the boundary are assigned zero (they are preserved by the derivative of f, hence have angle $2n\pi$ for $n \in \mathbb{Z}$ – we choose $n = 0$) and we require the choice be continuous in A (hence some angles might be larger than 2π, etc.). Now we divide Γ into two pieces

$$\Gamma_N = \{z \in \Gamma : \text{ the tangent vector } \underline{t}_z \text{ is rotated by less than } \pi \text{ by the derivative of } f \}.$$

$$\Gamma_R = \{z \in \Gamma : \text{ the tangent vector } \underline{t}_z \text{ is rotated by } \pi \text{ or more by the derivative of } f \}.$$

Then $\Gamma = \Gamma_N \cup \Gamma_R$ and $\Gamma_N \cap \Gamma_R = \emptyset$. (See Figure 4.)

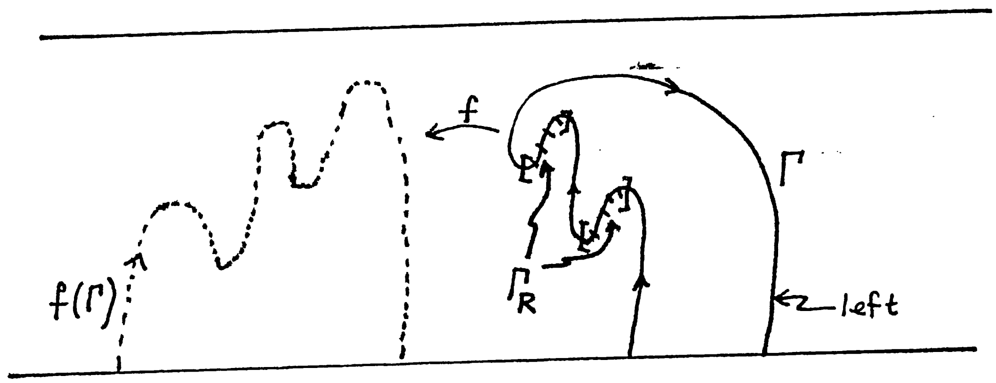

Figure 4.

Remark. We will see that the segments of Γ we use in the following construction are all contained in the "non rotating" part Γ_N of Γ.

Step II: The construction rules are the following: Rule (0) yields ℓ_0 while rules (1-3) yield ℓ_{n+1} given $\ell_0, \ldots, \ell_n$:

0) ℓ_0 is formed by choosing $z \in A \sim \Gamma$ within $\varepsilon > 0$ of the upper boundary of $A \sim \Gamma$ (where ε is chosen so that the rotation of all tangent vectors by Df is small) for points within ε of the boundary of A, and proceeding horizontally to the right if z is in an up region or left if z is in a down region of $A \sim \Gamma$ until a component of Γ (which must be in Γ_N) is encountered.

1) If n is even, so ℓ_n is a horizontal segment from end points z_0 in ℓ_{n-1} to z_1 in Γ, then ℓ_{n+1} is formed by following Γ from z_1 in the direction of the orientation assigned in step I until either a point of horizontal tangency is encountered or a "jump point" as described below in rule (2) is encountered.

2) If $n+1$ is odd, so ℓ_{n+1} is being formed by following a component of Γ as in rule (1), we identify two types of "jump points" as follows;

type 1: Suppose ℓ_{n+1} is on a left component of Γ, a point z of ℓ_{n+1} is called a jump point if there is a point $w \in \Gamma$ with

- •) w is on a left component of Γ,
- •) $\pi_y(z) = \pi_y(w), \pi_x(z) > \pi_x(w)$ and $\pi_x(f(z)) < \pi_x(f(w))$,
- •) if $\overline{zw}$ is the horizontal segment connecting z and w then $f(\overline{zw})$ is homotopic to $\overline{f(z)f(w)}$ rel $f(\ell_{n+1}) \cup \{f(w)\}$ (see Figure 5).

type 2: Suppose ℓ_{n+1} is on a right component of Γ, a point z of ℓ_{n+1} is called a jump point if there is a point $w \in \Gamma$ with

- •) w is on a right component of Γ,
- •) $\pi_y(z) = \pi_y(w)$, $\pi_x(z) < \pi_x(w)$ and $\pi_x(f(z)) > \pi_x(f(w))$,

•) if $\overline{zw}$ is the horizontal segment connecting z and w then $f(\overline{zw})$ is homotopic to $\overline{f(z)f(w)}$ rel $f(\ell_{n+1}) \cup \{f(w)\}$, (see Figure 5).

If while constructing ℓ_{n+1} a jump point z is encountered then ℓ_{n+1} is continued by the segment $\overline{zw}$ then following Γ from w in the direction of the orientation of step I (then return to rule (1)). (see Figure 6).

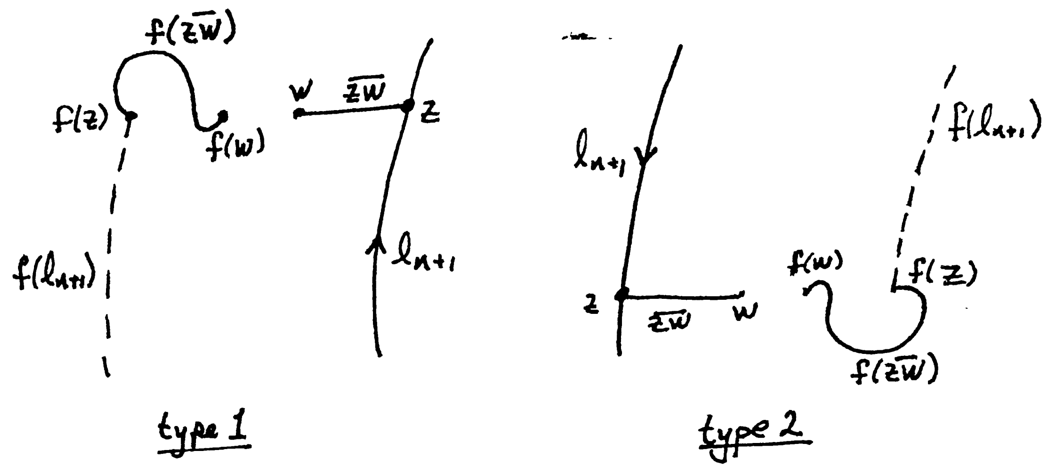

Figure 5.

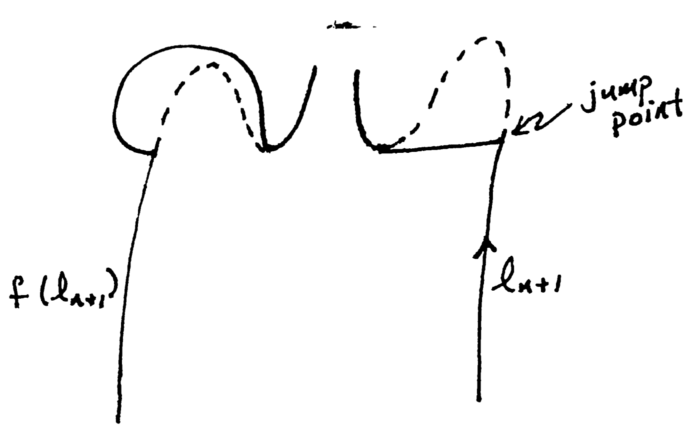

Figure 6.

3) If n is odd, so ℓ_n is made up of segments of Γ and horizontal segments from jump points and ℓ_n ends at a point z_0 of horizontal tangency of Γ, then ℓ_{n+1} is formed by the

longest horizontal segment beginning at z_0 in the direction of the tangent to Γ at z_0 so that

i) $\ell_{n+1} - f(\ell_{n+1})$ has positive index,

ii) both end points of Γ are in Γ_N,

iii) neither end point is a jump point (as in rule (2)),

iv) segments of ℓ_{n+1} in $A \sim \Gamma$ nearest the end points are either both in up or both in down components of $A \sim \Gamma$,

We give the following examples in Figure 7. These should be compared with the figures in Poincaré [P], however, we warn the reader that the labelling of Γ is different here.

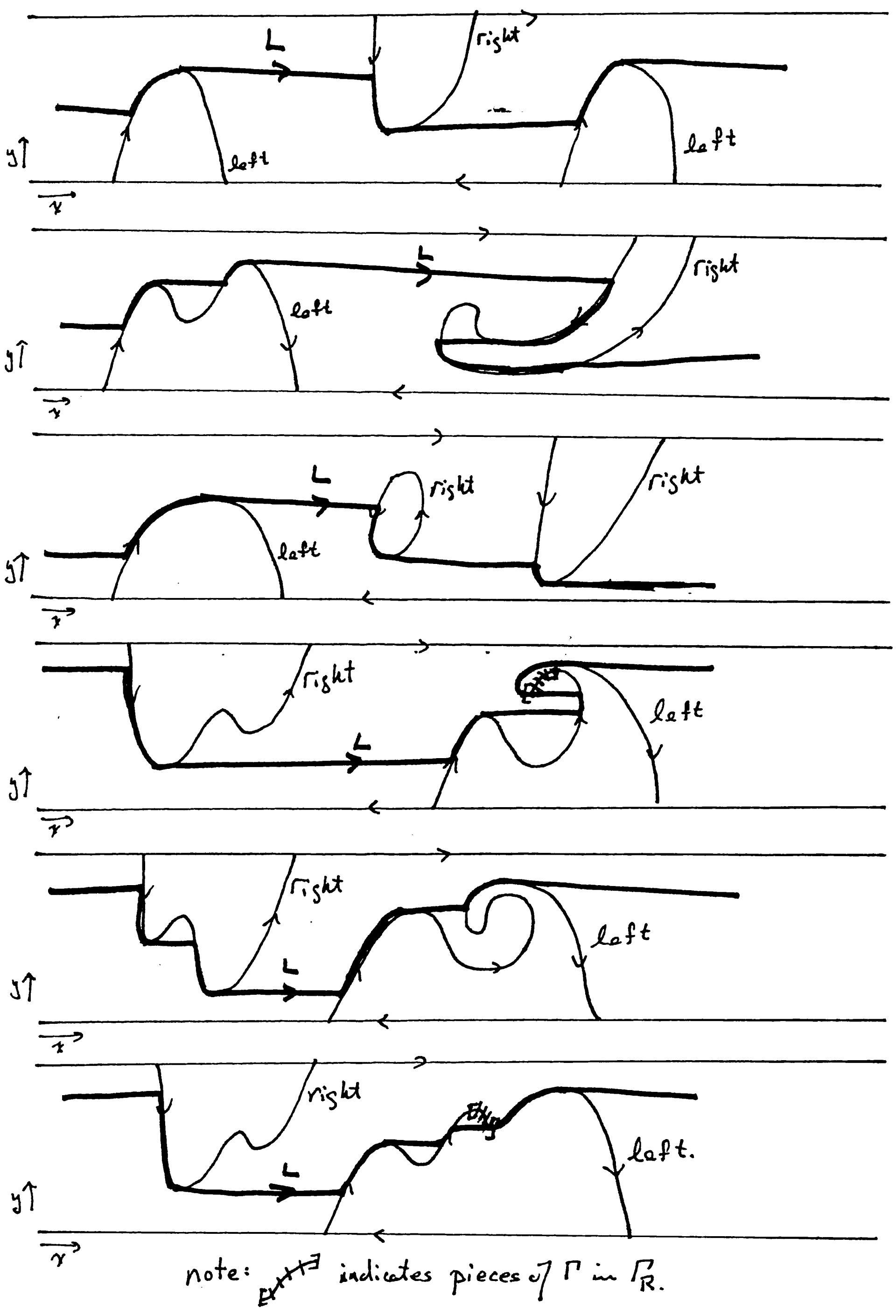

Figure 7.

Step III: Next we must show that the rules above produce the desired curve. That is, we must show that there is always a horizontal segment satisfying the conditions of rule (3) and that the curves ℓ created by rules (1) and (2) have $\ell - f(\ell)$ positive index. We begin with two lemmas giving examples of positive index arcs.

LEMMA 5. *Suppose ℓ is a horizontal segment with end points $z_1, z_2 \in \Gamma$ and interior in $A \sim \Gamma$ oriented from z_1 to z_2. Then if the signs of $\pi_x(z_1) - \pi_x(z_2)$ and $\pi_x(f(z_1)) - \pi_x(f(z_2)$ agree and ℓ is in an up component of $A \sim \Gamma$ if $\pi_x(z_1) < \pi_x(z_2)$ or ℓ is in a down component of $A \sim \Gamma$ if $\pi_x(z_2) < \pi_x(z_1)$ then $\ell - f(\ell)$ has positive index.*

More generally, let γ be a curve in A satisfying the following:

(1) $\pi_y\gamma(0) = \pi_y\gamma(1)$ *and* $\pi_x\gamma(1) > \pi_x\gamma(0)$

(2) $\pi_y(\varepsilon) > \pi_y\gamma(0)$

(3) *γ is homotopic to the segment $\ell = \overline{\gamma(0)\gamma(1)}$ rel $\{\gamma(0), \gamma(1)\}$*

(4) *The interior of the lift of γ in the universal covering of $A \sim \{\gamma(0), \gamma(1)\}$ does not intersect the lift of $\overline{\gamma(0)\gamma(1)}$ to which it is homotopic.*

Then the curve $\ell - \gamma$ has positive index.

Remark. The types of curves γ allowed by the second portion of the lemma are precisely those specified by requiring $\ell - \gamma$ to be positive index. They are those for which the intersections of ℓ with γ occur with non-zero winding (see Figure 8).

This lemma seems to be a particular case of a more general property of Jordan curves, where one would consider whether such a curve self intersects "from inside" or "from outside". "Outside" intersections add index, "inside" ones can give negative index. We can tell the "inside" intersection from the "outside" by looking at lifted curves in te appropriate covering spaces as in the statement of the lemma.

Proof. This follows recalling that positive index is counterclockwise (see Figure 8). □

LEMMA 6. *Suppose $\ell \subseteq \Gamma$ is an arc with end points z_1 and z_2 oriented from z_1 to z_2 such that the interior of ℓ contains no points of horizontal tangency of Γ. If ℓ is on a left component of Γ and $\pi_y(z_1) < \pi_y(z_2)$ or if ℓ is on a right component of Γ and $\pi_y(z_1) > \pi_y(z_2)$ then $\ell - f(\ell)$ has positive index.*

Proof. Again, follows from the choice of orientation (see Figure 9.). □

In the next 3 lemmas, we show that with two additional assumptions, there will be horizontal segments satisfying the conditions of rule (3). Rules (3) i and iv are treated in lemma 7, rule ii (and more) in lemma 8 and lemma 9 deals with iii.

LEMMA 7. *Suppose n is odd so that ℓ_n contains components of Γ and ends at a point z_0 of horizontal tangency of Γ. Suppose also that*

a) *$z_0 \in \Gamma_N$, the non-rotating part of Γ.*

b) *if the tangent to Γ at z_0 points right then it points into an up region of $A \sim \Gamma$, if left then it points into a down region of $A \sim \Gamma$.*

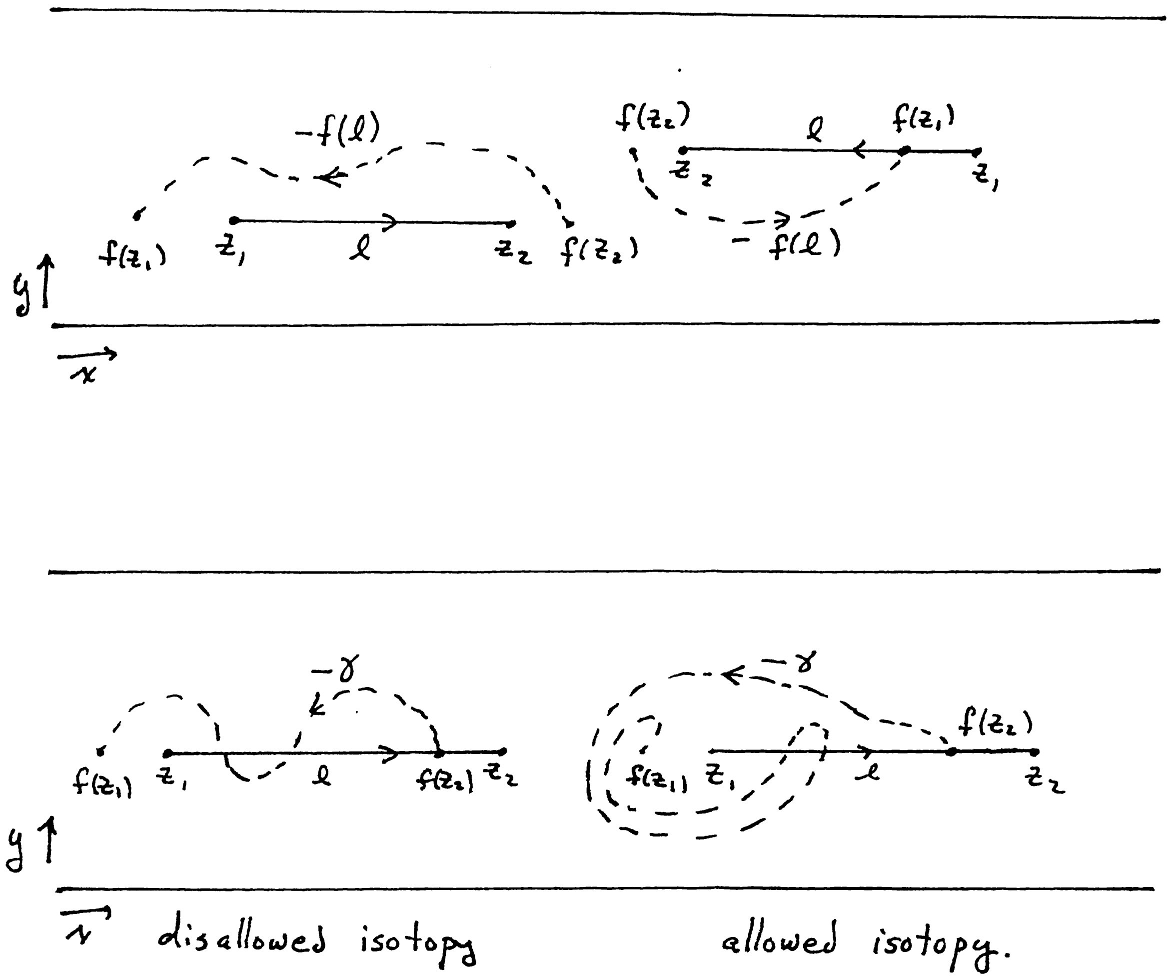

Figure 8.

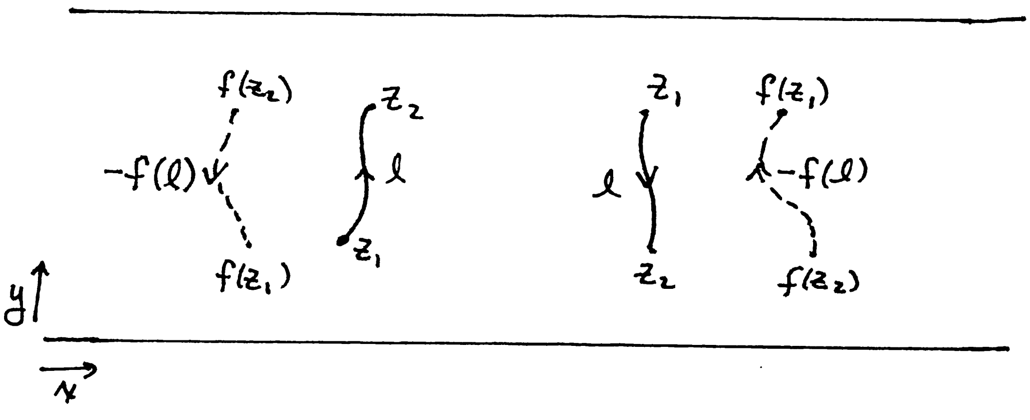

Figure 9.

Then there is a horizontal segment ℓ_{n+1} satisfying rule (3) i and iv.

Proof. Suppose the tangent to Γ at z_0 points to the right. Let R_0 be the horizontal ray beginning at z_0 and extending to the right and let R_1 be the horizontal ray starting at $f(z_0)$ and extending to the right.

Claim . By condition (a) of the Lemma, $f(R_0)$ will be homotopic to R_1 rel $f(\ell_n)$.

Proof of claim. (see Figure 10). If we form the suspension of f in $A \times [0,1]$ then we see that the points to the far right must stay to the right of ℓ_n and $f(\ell_n)$. Since the tangent vector to Γ at z_0 does not rotate and the image of R_0 must be as in the claim. □

Since $f(R_0)$ must cross R_1 infinitely many times,

there must be a point $f(z_1)$ in $f(R_0) \cap R_1$ such that the points to the left of z_1 are in an up component and such that $f(\overline{z_0 z_1})$ is homotopic to $\overline{f(z_0)f(z_1)}$ rel $f(\ell_n) \cup f(z_1)$.

condition (b) and lemma 5 complete the proof (See Figure 10.). The other case is symmetric. □

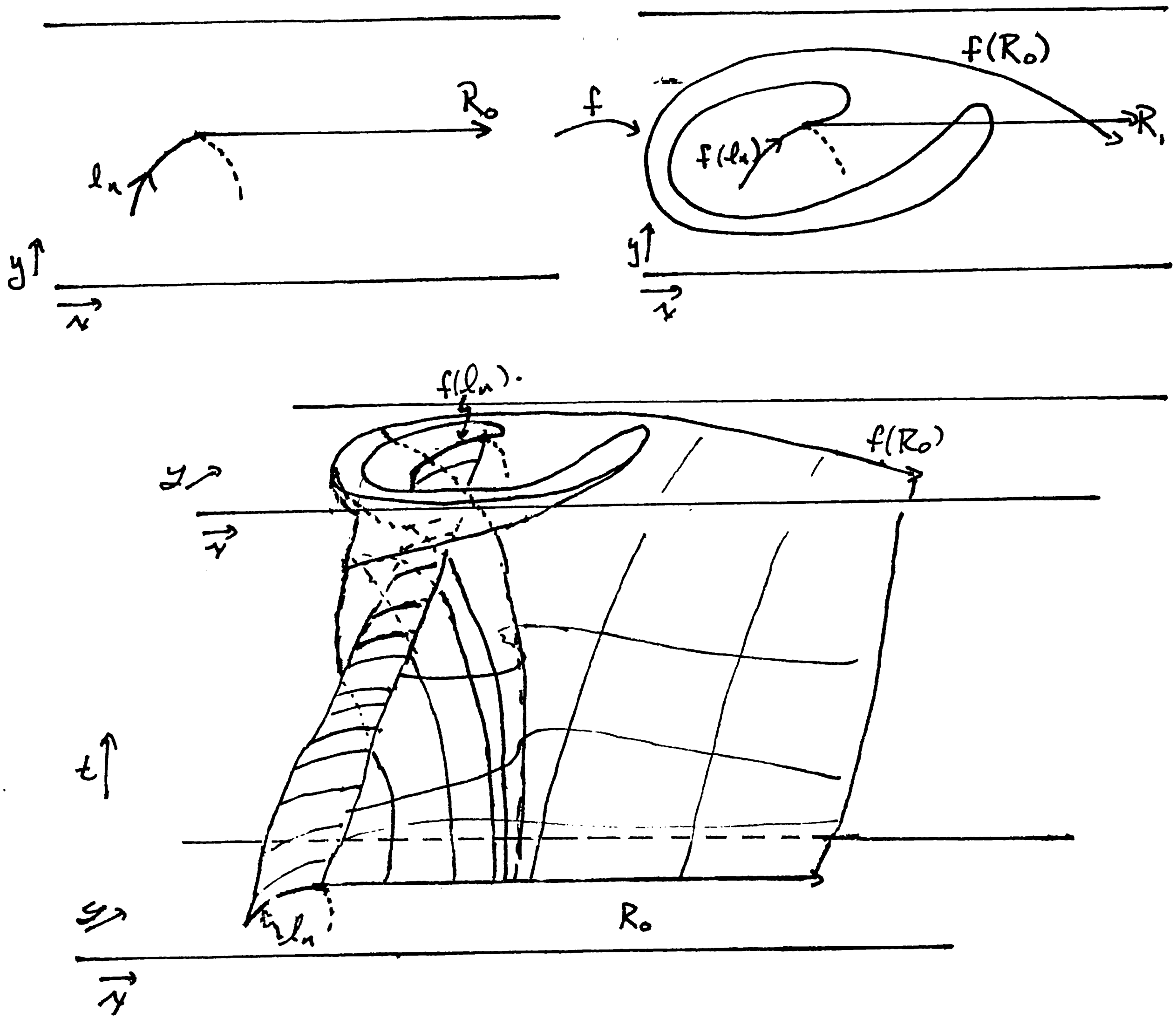

Figure 10.

Next we consider the type of components of Γ on which ℓ_{n+1} of the above lemma can end. In particular, we prove that ℓ_{n+1} satisfies condition ii of rule 3).

Lemma 8. *Suppose n is odd and ℓ_n satisfies conditions a and b of lemma 7, then the end point $z_1 \in \Gamma$ of ℓ_{n+1} which is not in ℓ_n will be in Γ_N and if it is on a left (respectively, right) component of Γ then the tangent to Γ at z_1 will point up (respectively, down), i.e., will have positive, (respectively, negative) y component.*

Proof. Since the segments of ℓ_{n+1} nearest the end points are either both in up region if ℓ_{n+1} is oriented to the right or both in down regions if it is oriented to the left, the rules of orientation of Γ imply that left (respectively, right) component of Γ at the end of ℓ_{n+1} will be oriented upward (respectively, downward).

Since the beginning point of ℓ_{n+1} is in Γ_N by assumption, it follows that the end point must also be in Γ_N, since ending at a point in Γ_R would not satisfy $\ell_{n+1} - f(\ell_{n+1})$ having positive index, (see Figure 11). □

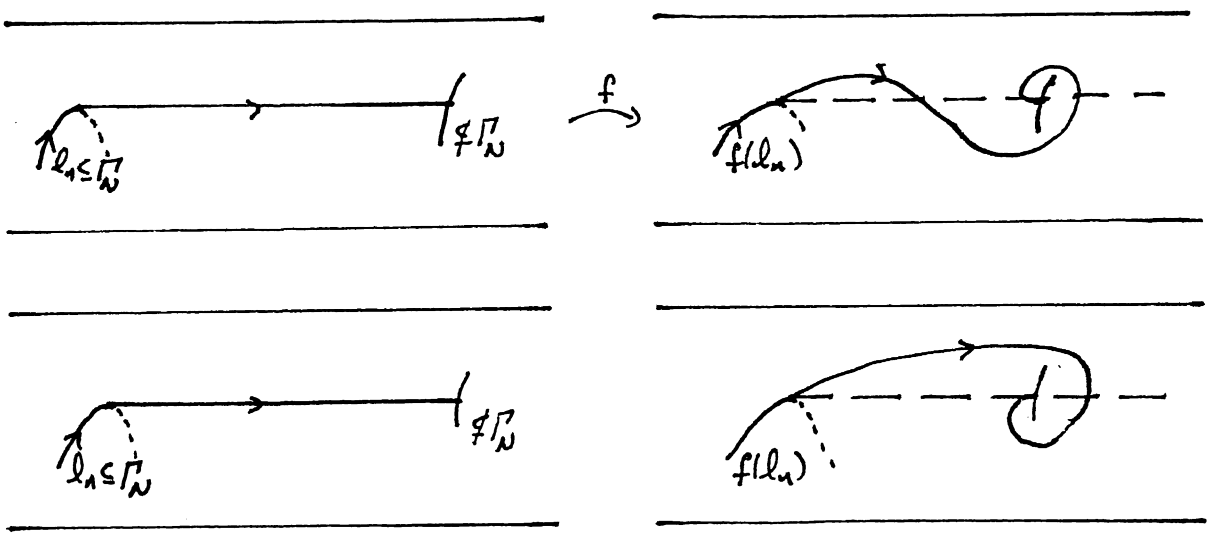

Figure 11. In both cases, ℓ_{n+1} must end in Γ_R not in Γ_R.

Finally, we must show that there is always a choice of ℓ_{n+1} satisfying the conditions of rule (3)iii.

LEMMA 9. *For n odd, assuming ℓ_n satisfies conditions a and b of lemma 7, there exists a horizontal segment ℓ_{n+1} satisfying the conditions of rule (3).*

Proof. Let a be the end point of ℓ_n in Γ and R_0 the horizontal ray pointing in the direction of the tangent at a. We assume a is in a left component of Γ. Then since f preserves orientation, there will always be horizontal segments ℓ_{n+1} satisfying i,ii, and iv of rule 3 as was found in lemma 7 and 8. We note that such an ℓ_{n+1} will have $f(\ell_{n+1})$ homotopic to the segment on the right pointing ray connecting the end points of $f(\ell_{n+1})$ fixing the end points, rel $f(\ell_n)$. If such a segment is chosen and it does not satisfy condition iii, then it may be extended to satisfy all of i-iv when R_0 points to the right. If R_0 points to the left then either ℓ_{n+1} may be extended to satisfy i-iv or the point a is already a jump point. (See Figure 12). □

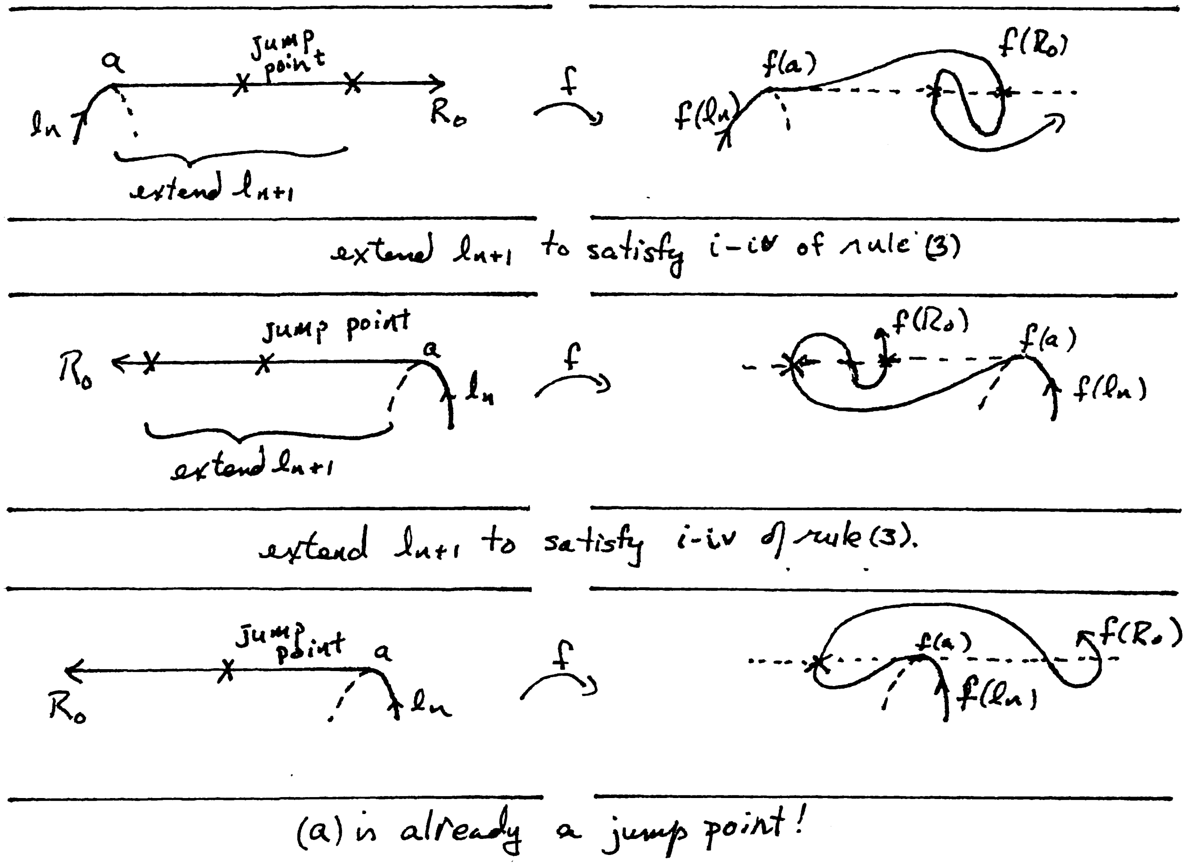

Figure 12.

Next we show that for n odd, the arcs ℓ_n will end at points of Γ_N. (assumption (a) of lemma 7).

LEMMA 10. *Suppose n is odd and that ℓ_n has its last segment with interior in Γ_N and is either upward oriented left or downward oriented right component of Γ. Then the end point of ℓ_n at a point of horizontal tangency of Γ is in Γ_N.*

Proof. Suppose not. We assume ℓ_n is in an upward oriented left component of Γ. Then the local picture near the end of ℓ_n is as in Figure 13.

Now case (a) does not occur because the component of $A \sim \Gamma$ to the left and above ℓ_n must be up. Hence we need only consider case (b). Now we consider the continuation of Γ after ℓ_n. Since its image is trapped by the image of the horizontal ray, we see we must have Γ again crossing the line horizontally tangent to Γ as in Figure 14.

In either case, points before the end point of ℓ_n are jump points. Hence rule (2) would apply before the point of horizontal tangency was reached. The case where ℓ_n is on a downward oriented right component of Γ is symmetric. ∎

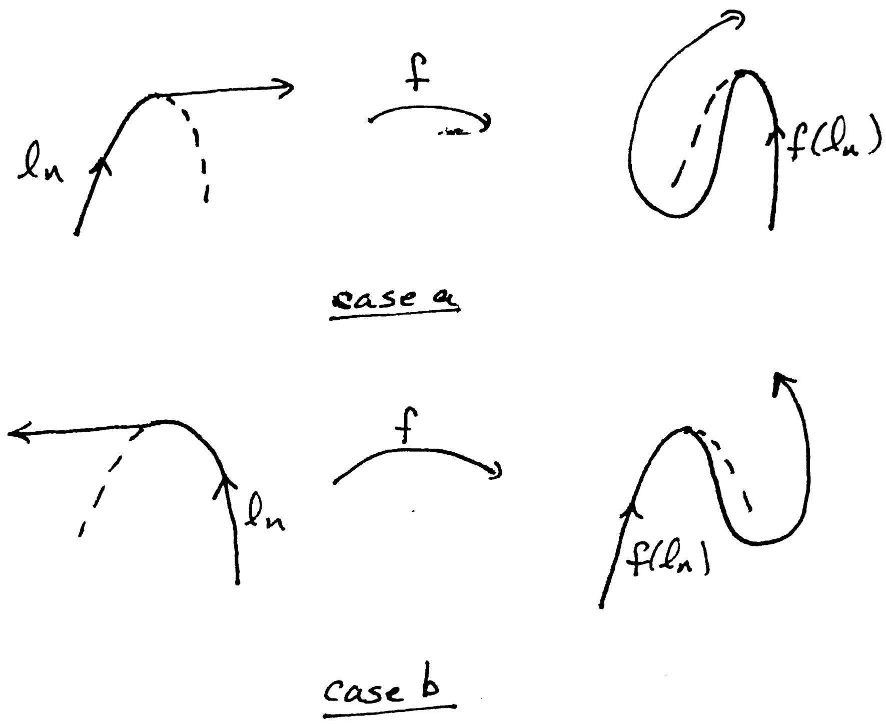

Figure 13.

Finally we must study the jump points, particularly the horizontal segments from a jump point and the character of the component of Γ on which it ends.

LEMMA 11. *Suppose n is odd so that ℓ_n is being formed by following Γ. Suppose z is a jump point of ℓ_n and the component of Γ in $\ell_n \cap \Gamma$ ending at z is either upward oriented left or downward oriented right in Γ_N. Let w be the point to which z jumps (as in rule (2)). Then*

1) *w is a local* min *of Γ if z is a left point or a local* max *of Γ if z is a right point,*
2) *the component of Γ which is connected to w in the direction of the tangent to w is upward oriented left if z is left, downward oriented right if z is right and is in Γ_N,*
3) *The segment $\overline{zw}$ is positive index (i.e. $\overline{zw} - f(\overline{zw})$ is a positive index loop).*

Proof. Condition (1) follows because ℓ_n does not begin at a jump point (see lemma 9), hence jump points arrive only when w is a point of horizontal tangency of Γ.

Condition (2) follows by the choice of orientation in step I and the fact that any rotation of the points in the same component of Γ as w and near w would result in jump points on ℓ_n before z.

Condition (3) follows exactly as lemma (5). ▯

Step IV: To complete the proof we note the following:

•) for n odd, the portion of ℓ_n contained Γ are all in Γ_N (rule (3) and lemma 8, 11).

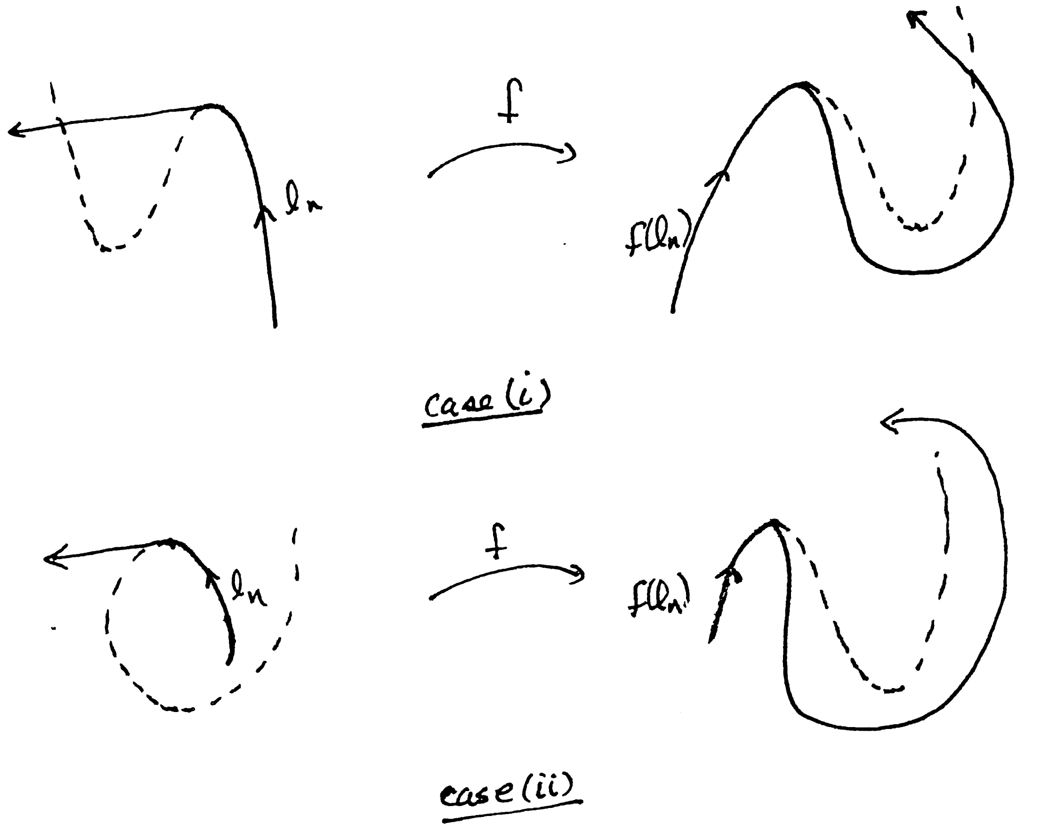

Figure 14.

- •) for n odd, if ℓ_n is in left components of Γ then it is oriented upwards, if ℓ_n is n right components of Γ then it is oriented downwards (lemmas 7,8,10).
- •) $\ell_n - f(\ell_n)$ is always positive index (rule (3) and lemmas 5,6,7,8,10,11)
- •) if we construct the segments $\ell_0, \ldots \ell_n, \ldots$ as above, eventually a closed loop is formed (lemma (3) genericity assumption on Γ).

This closed loop is the required loop L which by lemma 2 completes the proof for maps satisfying the genericity assumptions of lemma 3. Again noting that the limit of maps on with a fixed point will have a fixed point, we obtain the desired result.

Concluding remarks. 1) The second theorem of Birkhoff actually yields two periodic orbits. Poincaré conjectures existence of two fixed points, but he only discusses the case of "generic" Γ. We do not know if there is a direct proof of existence of the second fixed point using these techniques.

2) It would also be interesting to know what Γ looks like for a generic twist map. Computer studies show it can be quite complicated, even for simple maps.

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