SMOOTH INVARIANT FOLIATIONS IN INFINITE DIMENSIONAL SPACES

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§1. Introduction.

One of the most useful properties of dynamical systems is the existence of invariant manifolds and their invariant foliations near an equilibrium or a periodic orbits. These manifolds and foliations serve as a convenient setting to describe the qualitative behavior of the local flows, and in many cases they are useful tools for technical estimates which facilitate the study of the local bifurcation diagram (see [6]). Many other important concepts in dynamical systems are closely related to the invariant manifolds and foliations. In finite dimensional space, the relations among invariant manifolds, invariant foliations, λ -lemma, linearization and homoclinic bifurcation have been studied in [11]. It is well known that if each leaf is used as a coordinate, the original systems is completely decoupled and the linerization follows easily (for example, see [27] and [22]).

As a motivation, let us consider a linear system in $\mathbf{R}^{m+n}$

$$\dot{u} = Au, \qquad u \in \mathbf{R}^m$$

$$\dot{v} = Bv, \qquad v \in \mathbf{R}^n$$

with $\mathrm{Re}\sigma(A) > \gamma > \mathrm{Re}\sigma(B)$, where A and B are matrices, $\sigma(A)$ and $\sigma(B)$ are spectra of A and B with Re denoting the real parts and $\gamma \in \mathbf{R}$ is a constant. For a given $u_0 \in \mathbf{R}^m$, after $t > 0$ the n-dimensional submanifold

$$M_0 = \{(u,v) | u = u_0, v \in \mathbf{R}^n\}$$

is carried by the flow to a new submanifold

$$M_t = \{(u,v) | u = e^{At}u_0, v \in \mathbf{R}^n\}.$$

Moreover, if (u_1, v_1) and $(u_2, v_2) \in M_0$, then

$$||(e^{At}u_1, e^{Bt}v_1) - (e^{At}u_2, e^{Bt}v_2)|| = O(e^{\gamma t}), \quad \text{as}\, t \to +\infty,$$

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while points not in M_0 depart more rapidly than $Ce^{\gamma t}$, as $t \to +\infty$. Thus, we are able to group points in $\mathbf{R}^{m+n}$ as equivalent classes according to their asymptotic behavior as $t \to +\infty$, and each asymptotic class is a submanifold $u =$ constant. We expect that these observation will persist after the adding of small nonlinear terms.

Let X be a Banach space, $\gamma \in \mathbf{R}$ and $T(\cdot,\cdot) : X \times \mathbf{R}^+ \to X$ be a nonlinear semigroup. We say that W^s_γ is a γ-stable fiber if $|T(x_1,t) - T(x_2,t)| = O(e^{-\gamma t})$ as $t \to +\infty$ for any x_1, $x_2 \in W^s_\gamma$. We use $W^s_\gamma(x)$ to denote a γ-stable fiber passing through $x \in X$.

Let $Y \subset X$ be such that the backward flow $T(.,.); Y \times \mathbf{R}^+ \to Y$ is uniquely defined. We say that $W^u_\gamma(y)$ is a γ-unstable fiber passing through $y \in Y$ if $|T(y_1,t) - T(y_2,t)| = O(e^{-\gamma t})$ as $t \to -\infty$ for any y_1, $y_2 \in W^u_\gamma$.

If W^s_γ is an invariant manifold, we say W^s_γ is a γ-stable manifold. Similarly, we have γ-unstable manifolds. It follows from the definition that γ-stable (respt., γ-unstable) fibers are invariant under the forward (respt., backward) flow T, i.e.:

$$T(W^s_\gamma(x), t) \subset W^s_\gamma(T(x,t)), \text{ for } t \in \mathbf{R}^+,$$

$$T(W^u_\gamma(y), t) \subset W^u_\gamma(T(y,t)), \text{ for } t \in \mathbf{R}^-.$$

The purpose of this paper is to show that for some dynamical systems generated by partial differential equations, $W^s_\gamma(x)$ and $W^u_\gamma(y)$ are manifolds and to study the smoothness of those manifolds. We are also interested in the smooth dependence of $W^s_\gamma(x)$ (or $W^u_\gamma(y)$) on x (or y).

The smoothness of the invariant foliations suffers from two restrictions. First, if the nonlinear term in the equation is C^k, each fiber can only be a C^r submanifold with $r \leq k$. Secondly, there is a gap condition which requires that a gap between the real part of the spectrum of the linear equation has to be large compared with the module of the nonlinear term. The examples given in [9] show that if the gap conditions fail, then the invariant manifolds lose smoothness. It is well known that these gap conditions are always satisfied in the study of center, center-stable and center-unstable manifolds (see [3], [5], [6], [7], [20], [19],[21], [24] and [30]). Thus we are able to obtain the same smoothness, i.e., $r = k$. See, for example, [6], [7], [30], and [31]. The theorems we will give are closely related to the theory of inertial manifolds and generalize some recent results. For the theory of inertial manifolds, see [9], [10], [13], [15], [16], [17], [22], [25], [26], also see books Hale [18], Temám[29], and their references.

More delicate is the smooth dependence of $W^s_\gamma(x)$ (or $W^u_\gamma(y)$) with respect to x (or y). Here we do not have C^k dependence on x (or y) even if the vector field is C^k, otherwise we would have obtained a C^k linearization theorem which is in general not true (see [28]). A general condition will be given in this paper which is similar to the gap condition we mentioned before. Accordingly, we can prove under very general assumptions $W^s_\gamma(x)$ is Hölder continuous in x. And in some special cases, such as $\mathrm{Re}\sigma(A) \leq 0$, then $W^s_\gamma(x)$ is

C^{k-1} with respect to x. In general we do not have a C^{k-1} foliation of whole space but we do have a C^{k-1} foliation on the center-stable manifold W^{cs} and the center-unstable manifold W^{cu}. For application of this fact see [6] and [12].

There have been some geometric proofs of the invariant foliations ine finite dimensional spaces, which are based on the concept of graph transforms, see [2], [14] and [21] for example. Ours is an analytic proof which is based on the variation of constants formula (i.e., Liapunov-Perron formula) and generalized exponential dichotomies for semiflows in infinite dimensional spaces (see [20]). After an integral equation is written, the smallness of the nonlinear term usually guarantees the existence of a fixed point of the derived mapping by contraction mapping theorem. The smoothness of the fixed point with respect to the parameters is then studied from the integral equation. This allows a unified treatment of the whole problems.

We introduce the main notations and definitions in Section 2. Section 3 contains some basic theorems and lemmas which will be used throughout the paper. A study of the abstract parabolic evolution equations is given in Section 4.

§2 Notations.

Let X and Y be Banach spaces and $U \subset X$ be an open subset. We define the following Banach spaces

(1) For any integer $k > 0$, define the Banach space

$$C^k(U,Y) = \{f | f : U \to Y \text{ is } k-\text{times differentiable and } D^i f \text{ is bounded and continuous for } 0 \le i \le k\}$$

with the norm $||f||_k = \sum_{i=0}^k \sup_{x \in U} |D^i f(x)|_Y$, where D is the differential operator.

(2) Let $0 < \alpha \le 1$, we define the Banach space

$$C^{k,\alpha}(U,Y) = \{f \in C^k(U,Y) || D^k f|_\alpha = \sup_{x_1 \neq x_2} \frac{|D^k f(x_1) - D^k f(x_2)|_Y}{|x_1 - x_2|_X^\alpha} < \infty\}$$

with the norm $|f|_{k,\alpha} = |f|_k + |D^k f|_\alpha$. For simplicity, we will write C^α for $C^{0,\alpha}$.

(3) Let $\gamma \in \mathbf{R}$ and $\tau \in \mathbf{R}$ be fixed. We define the Banach space

$$E_\tau^-(\gamma, X) = \{f : (-\infty, \tau] \to X \text{ is continuous and } \sup_{t \in (-\infty,\tau]} |e^{\gamma t} f(t)|_X < \infty\}$$

with the norm $|f|_{E_\tau^-(\gamma,X)} = \sup_{t \in (-\infty,\tau]} |e^{\gamma t} f(t)|_X$,

(4) Similarly, we define the Banach space

$$E_\tau^+(\gamma, X) = \{f : [\tau, \infty) \to X \text{ is continuous and } \sup_{t \in [\tau,\infty)} |e^{\gamma t} f(t)|_X < \infty\}$$

with the norm $|f|_{E_\tau^+(\gamma,X)} = \sup_{t\in[\tau,\infty)} |e^{\gamma t} f(t)|_X$,

(5) We use $L^k(X,Y), k > 0$, to denote the Banach space of all k-linear maps from X to Y with the norm $|\cdot|_{L^k(X,Y)}$.

(6) Let $n > k > 0$ be integers and Λ be an index set. Let M^n be a n-dimensional manifold and M_λ^k, $\lambda \in \Lambda$, be k-dimensional submanifolds of M^n. We say that M^n has a C^r foliation indexed by $\lambda \in \Lambda$ if $M^n = \cup_{\lambda\in\Lambda} M_\lambda^k$ and M_λ^k are mutually disjoint. Each M_λ^k is called a leaf through $\lambda \in \Lambda$ and is a injectively immersed connected submanifold. Moreover, M^n is covered by C^r chart $\phi : D^k \times D^{n-k} \to M^n$ with $\phi(D^k \times y) \subset M_\lambda^k$, where $\phi(0, y) \in M_\lambda^k$ and D^s is the unit s-dimensional disk. Let $\pi(t, x)$, $t > 0$ and $x \in M^n$ be a semiflow on M^n. The foliation $M^n = \cup_{\lambda\in\Lambda} M_\lambda^k$ is said to be invariant under π if $\pi(t, M_\lambda^k)$ is contained in a leaf for every $t \geq 0$.

§3 Main Results.

Let X, Y and Z be Banach spaces. Assume that $X \subset Y \subset Z$, X is continuously imbedded in Y and Y is continuously imbedded in Z. Let $T(t,s)$ be an evolution operator on Z, which means that $T(t,s) \in L(Z,Z)$ $(t \geq s)$ is defined on interval $J \subset \mathbb{R}$; ordinarily $J = \mathbb{R}$ or $[\tau,\infty)$ or $(-\infty,\tau]$ and satisfies : (a) $T(t,t) = I = identity$; (b) $T(t,s)T(s,r) = T(t,r)$ if $t \geq s \geq r$; (c) $T(t,s)$ is strongly continuous in (t,s).

We say that $T(t,s)$ has a pseudo-dichotomy on the triplet (X,Y,Z), or on Z for short, if there exist continuous projection $P(t), t \in J$ and constants $\alpha, \beta > 0, \alpha < \beta, 0 \leq \rho < 1$ and $M_i > 0, i = 1,2,3,4$, such that

(i) $T(t,s)P(s) = P(t)T(t,s), t \geq s$, $T(t,s)Y \subset X, t > s$, and $R(P(t)) \subset X$, where $R(P(t))$ denotes the range of the operator $P(t)$,

(ii) the restriction $T(t,s)|_{R(P(s))}, t \geq s$ is an isomorphism from $R(P(s))$ onto $R(P(t))$, and we define $T(s,t)$ as the inverse map from $R(P(t))$ to $R(P(s))$,

(iii) the following equalities hold

$$|T(t,s)P(s)x|_X \leq M_1 e^{-\alpha(t-s)}|x|_X, \quad \text{for } t \leq s \tag{3.1}$$

$$|T(t,s)P(s)y|_X \leq M_2 e^{-\alpha(t-s)}|y|_Y, \quad \text{for } t \leq s \tag{3.2}$$

$$|T(t,s)(I - P(s))x|_X \leq M_3 e^{-\beta(t-s)}|x|_X, \quad \text{for } t \geq s \tag{3.3}$$

$$|T(t,s)(I - P(s))y|_X \leq M_4 (t-s)^{-\rho} e^{-\beta(t-s)}|y|_Y, \quad \text{for } t > s. \tag{3.4}$$

REMARK. *Condition (3.4) is a smooth property of the evolution operator $T(t,s)$. Condition (ii) is not very restrictive since in many cases the unstable space is finite dimensional.*

Let $\tau \in \mathbb{R}$ and $J = (-\infty, \tau]$. Define an operator L as

$$(Lf)(t) = \int_\tau^t T(t,s)P(s)f(s)ds + \int_{-\infty}^t T(t,s)Q(s)f(s)ds, \tag{3.5}$$

where $f \in E_\tau^-(\gamma, Y), \alpha < \gamma < \beta$, and $Q(s) = I - P(s)$.

LEMMA 3.1. *If $T(t,s)$ has a pseudo-dichotomy on Z, then the operator L defined by (3.5) is a bounded linear operator from $E_\tau^-(\gamma,Y)$ to $E_\tau^-(\gamma,X)$ and the norm of L satisfies the following estimate*

$$||L|| \leq K(\beta-\gamma,\gamma-\alpha,\rho),$$

where $K:(0,\infty)\times(0,\infty)\times[0,1)\to\mathbb{R}^+$ is defined by

$$K(a,b,c) = M_2 b^{-1} + M_4\frac{2-c}{1-c}a^{c-1}. \tag{3.6}$$

Proof. By using (3.2) and (3.4), we have that

$$\begin{aligned}|Lf|_{E_\tau^-(\gamma,X)} &\leq \sup_{t\in(-\infty,\tau]}\{\int_t^\tau e^{\gamma t}|T(t,s)P(s)f(s)|_X ds + \int_{-\infty}^t e^{\gamma t}|T(t,s)Q(s)f(s)|_X ds\}\\ &\leq M_2(\gamma-\alpha)^{-1} + M_4\frac{2-\rho}{1-\rho}(\beta-\gamma)^{\rho-1}.\end{aligned}$$

This completes this proof. □

LEMMA 3.2. *Suppose that $\alpha<\gamma<\beta$, $u:(-\infty,\tau]\to[0,\infty)$ is continuous, $\sup t \leq \tau e^{\gamma t}u(t) < \infty$ and satisfies for $t\leq\tau$*

$$u(t) \leq C_1 e^{-\alpha(t-\tau)} + C_2\int_\tau^t e^{-\alpha(t-s)}u(s)ds + C_3\int_{-\infty}^t (t-s)^{-\rho}e^{-\beta(t-s)}u(s)ds, \tag{3.7}$$

where C_1, C_2 and C_3 are positive constants satisfying

$$C_2(\gamma-\alpha)^{-1} + C_3\frac{2-\rho}{1-\rho}(\beta-\gamma)^{\rho-1} < 1. \tag{3.8}$$

Then

$$u(t) \leq (1 - C_2(\gamma_1-\alpha)^{-1} - C_3\frac{2-\rho}{1-\rho}(\beta-\gamma_1)^{\rho-1})^{-1}C_1 e^{-\gamma_1 t} \tag{3.9}$$

for any γ_1, $\alpha<\gamma_1<\beta$, satisfying

$$C_2(\gamma_1-\alpha)^{-1} + C_3\frac{2-\rho}{1-\rho}(\beta-\gamma_1)^{\rho-1} < 1.$$

Proof. Without losing generality, we assume $\tau=0$. We will prove that if $v(t)$, $t\leq 0$, satisfies that

$$v(t) = C_1 e^{-\alpha(t-\tau)} + C_2\int_\tau^t e^{-\alpha(t-s)}v(s)ds + C_3\int_{-\infty}^t (t-s)^{-\rho}e^{-\beta(t-s)}v(s)ds \tag{3.10}$$

and $\sup_{t\le 0}|e^{\gamma t}v(t)| < \infty$, then

$$0 \le v(t) \le (1 - C_2(\gamma_1-\alpha)^{-1} - C_3\frac{2-\rho}{1-\rho}(\beta-\gamma_1)^{\rho-1})^{-1}C_1e^{-\gamma_1 t}$$

and $u(t) \le v(t)$ for $t \le 0$.

By using the contraction mapping theorem and (3.8), we have that (3.10) has a unique solution $v(t)$ satisfying $\sup_{t\le 0}|e^{\gamma t}v(t)| < \infty$. If $v(t) < 0$ for some $t = t_0$, then $\inf_{t\le 0}\{e^{\gamma t}v(t)\} < 0$. Hence

$$e^{\gamma t}v(t) \ge C_1e^{(\gamma-\alpha)t} + (C_2(\gamma-\alpha)^{-1} + C_3\frac{2-\rho}{1-\rho}(\beta-\gamma)^{\rho-1})\inf_{t\le 0} e^{\gamma t}v(t)$$

Thus $e^{\gamma t}v(t) \ge 0$. This is a contradiction and proves that $v(t) \ge 0$.

For γ_1, (3.10) has a unique solution $w(t)$ which satisfies

$$w(t) \le (1 - C_2(\gamma_1-\alpha)^{-1} - C_3\frac{2-\rho}{1-\rho}(\beta-\gamma_1)^{\rho-1})^{-1}C_1e^{-\gamma_1 t}.$$

By the uniqueness, we have that $v(t) = w(t)$.

Next observe that

$$\begin{aligned} u(t)-v(t) \le& C_2\int_\tau^t e^{-\alpha(t-s)}(u(s)-v(s))ds \\ &+ C_3\int_{-\infty}^t (t-s)^{-\rho}e^{-\beta(t-s)}(u(s)-v(s))ds. \end{aligned} \tag{3.11}$$

If $u(t) - v(t) > 0$ for some $t = t_0$, then

$$\sup_{t\le 0}\{e^{\gamma t}(u(t)-v(t))\} > 0.$$

From (3.11), we have that

$$e^{\gamma t}(u(t)-v(t)) \le (C_2(\gamma_1-\alpha)^{-1} + C_3\frac{2-\rho}{1-\rho}(\beta-\gamma_1)^{\rho-1})\sup_{t\le 0}\{e^{\gamma t}(u(t)-v(t))\}.$$

By (3.8), $e^{\gamma t}(u(t)-v(t)) \le 0$. This contradiction proves that $u(t) - v(t) > 0$ for some $t = t_0$ is impossible. This completes this proof. □

Let Λ be a Banach space. Let $\tau \in \mathbb{R}$ and J be either the interval $(-\infty, \tau]$ or the interval $[\tau, \infty)$. Consider the following nonlinear map

$$F : J \times X \times \Lambda \to Y.$$

We assume that the nonlinear operator F satisfies

Hypothesis A. *F is a continuous mapping from $J \times X \times \Lambda$ to Y with bounded continuous Frechét derivatives $D_u^{k_1} D_\lambda^{k_2} F(t,u,\lambda)$ with respect to u and λ, $k_1 + k_2 \leq k$, where k is a given positive integer.*

For the above nonlinear mapping F with $J = (-\infty, \tau]$, we consider the following nonlinear integral equation

$$
\begin{aligned}
u(t) = T(t,\tau)P(\tau)\xi + \int_\tau^t T(t,s)P(s)F(s,u(s),\lambda)ds \\
+ \int_{-\infty}^t T(t,s)Q(s)F(s,u(s),\lambda)ds,
\end{aligned}
\tag{3.12}
$$

where $\xi \in X, u \in E_\tau^-(\gamma, X)$ and $\lambda \in \Lambda$. We assume that $T(t,s)$ is an evolution operator with a pseudo-dichotomy on Z. It is not hard to see from the definition of pseudo-dichotomy and Hypothesis A that the right hand side of (3.12) is well defined.

Our first theorem is

Theorem 3.3. *Assume that the evolution operator $T(t,s)$ has a pseudo-dichotomy on Z and F satisfies the Hypothesis A with $J = (-\infty, \tau]$. If there is $\gamma > 0$ such that*

$$\alpha < \gamma \leq k\gamma < \beta, \tag{3.13}$$

$$K(\beta - k\gamma, \gamma - \alpha, \rho) Lip_u F < 1, \tag{3.14}$$

where $Lip_u F$ is the Lipschitz constant of F with respect to u, then we have that for every $(\xi, \lambda) \in X \times \Lambda$ the integral equation (3.12) has a unique solution $u(.;\xi,\lambda) \in E_\tau^-(\gamma, X)$ which has the following property:

$$u : X \times \Lambda \to E_\tau^-(k\gamma, X)$$

is a C^k mapping.

Remark. *The condition (3.13) describes the spectral gap. The examples in [9] imply that if the condition (3.13) fails, then the solutions of (3.12) will lose smoothness. We will apply this theorem to get invariant manifolds for evolutionary equations in the next section.*

Proof. Let $\mathcal{F}(u,\xi,\lambda)$ be the right hand side of (3.12), i.e.,

$$
\begin{aligned}
\mathcal{F}(u,\xi,\lambda) = T(t,\tau)P(\tau)\xi + \int_\tau^t T(t,s)P(s)F(s,u(s),\lambda)ds \\
+ \int_{-\infty}^t T(t,s)Q(s)F(s,u(s),\lambda)ds.
\end{aligned}
$$

From the definition of pseudo-dichotomy and the Hypothesis A on F with $J = (-\infty, \tau]$ we have that $\mathcal{F}$ is well defined from $E_\tau^-(\gamma, X) \times X \times \Lambda$ to $E_\tau^-(\gamma, X)$ and is continuous. Furthermore, for $\xi_1, \xi_2 \in X$ we have that

$$|\mathcal{F}(u, \xi_1, \lambda) - \mathcal{F}(u, \xi_2, \lambda)|_{E_\tau^-(\gamma, X)} \leq M_1 e^{\gamma\tau} |\xi_1 - \xi_2|_X.$$

For each $u, \bar{u} \in E_\tau^-(k\gamma, X)$, by using Lemma 3.1, we have that

$$\begin{aligned}
&|\mathcal{F}(u, \xi_1, \lambda) - \mathcal{F}(\bar{u}, \xi_2, \lambda)|_{E_\tau^-(\gamma, X)} \\
&\leq K(\beta - \gamma, \gamma - \alpha, \rho) Lip_u F |u - \bar{u}|_{E_\tau^-(\gamma, X)}.
\end{aligned}$$

Since $K(\beta - \gamma, \gamma - \alpha, \rho) Lip_u F < 1$ from the assumption of this theorem, we have that $\mathcal{F}$ is a uniform contraction with respect to the parameters ξ and λ. Using the uniform contraction principle, we have that for each $(\xi, \lambda) \in X \times \Lambda$ $\mathcal{F}(\cdot, \xi, \lambda)$ has a unique fixed point $u(\cdot; \xi, \lambda) \in E_\tau^-(\gamma, X)$ and $u(\cdot; \xi, \lambda)$ is jointly continuous in (ξ, λ) and is Lipschitz continuous in ξ. Moreover, we have that

$$|u(\cdot; \xi_1, \lambda) - u(\cdot; \xi_2, \lambda)|_{E_\tau^-(\gamma, X)} \leq \frac{M_1 e^{\gamma\tau}}{1 - K(\beta - \gamma, \gamma - \alpha, \rho) Lip_u F} |\xi_1 - \xi_2|_X. \tag{3.15}$$

In other words, $u(t; \xi, \lambda)$ is a solution of (3.12) which satisfies (3.15). Next we want to show that u is C^k from $X \times \Lambda$ to $E_\tau^-(k\gamma, X)$. We are going to prove this by induction on k. Here we should mention that the method which we are going to use to show the smoothness of the solution u is different from those used in [7], [8] and [9].

First let us consider $k = 1$. Since $K(\beta - \gamma, \gamma - \alpha, \rho) Lip_u F < 1$, there is a small $\delta > 0$ such that $\alpha < \gamma - 2\delta$ and

$$K(\beta - \gamma_1, \gamma_1 - \alpha, \rho) Lip_u F < 1, \text{ for } \gamma - 2\delta \leq \gamma_1 \leq \gamma. \tag{3.16}$$

Using Lemma 3.2, we have that $u(.; \xi, \lambda) \in E_\tau^-(\gamma_1, X)$ for $\gamma - 2\delta \leq \gamma_1 \leq \gamma$. Let

$$\begin{aligned}
I = \{ &\int_\tau^t T(t, s) P(s) [F(s, u(s; \xi_1, \lambda), \lambda) - F(s, u(s; \xi_2, \lambda), \lambda) \\
&\qquad - D_u F(s, u(s; \xi_2, \lambda), \lambda)(u(s; \xi_1, \lambda) - u(s; \xi_2, \lambda))] ds \\
+ &\int_{-\infty}^t T(t, s) Q(s) [F(s, u(s; \xi_1, \lambda), \lambda) - F(s, u(s; \xi_2, \lambda), \lambda) \\
&\qquad - D_u F(s, u(s; \xi_2, \lambda), \lambda)(u(s; \xi_1, \lambda) - u(s; \xi_2, \lambda))] ds \}.
\end{aligned}$$

We claim that $|I|_{E_\tau^-(\gamma - \delta, X)} = o(|\xi_1 - \xi_2|)$ as $\xi_1 \to \xi_2$. Using this claim, we have that

$$\begin{aligned}
&u(\cdot; \xi_1, \lambda) - u(\cdot; \xi_2, \lambda) \\
&- \int_\tau^t T(t, s) P(s) D_u F(s, u(s; \xi_2, \lambda), \lambda)(u(s; \xi_1, \lambda) - u(s; \xi_2, \lambda)) ds \\
&- \int_{-\infty}^t T(t, s) Q(s) D_u F(s, u(s; \xi_2, \lambda), \lambda)(u(s; \xi_1, \lambda) - u(s; \xi_2, \lambda)) ds \\
&= T(t, \tau) P(\tau)(\xi_1 - \xi_2) + I \\
&= \mathcal{J}(\xi_1 - \xi_2) + o(|\xi_1 - \xi_2|_X), \text{ as } \xi_1 \to \xi_2, \qquad (3.17)
\end{aligned}$$

where $\mathcal{J} = T(t,\tau)P(\tau)$ is a bounded linear operator from X to $E_\tau^-(\gamma-\delta, X)$. Let

$$\mathcal{L}f = \int_\tau^t T(t,s)P(s)D_uF(s,u(s;\xi_2,\lambda),\lambda)f ds + \int_{-\infty}^t T(t,s)Q(s)D_uF(s,u(s;\xi_2,\lambda),\lambda)f ds.$$

Using (3.16) and Lemma 3.1, we have that $\mathcal{L}$ is a continuous linear operator from $E_\tau^-(\gamma - \delta, X)$ to itself and $|\mathcal{L}|_{L(E_\tau^-(\gamma-\delta,X),E_\tau^-(\gamma-\delta,X))} < 1$. (3.17) implies that

$$u(\cdot;\xi_1,\lambda) - u(\cdot;\xi_2,\lambda) = \mathcal{L}^{-1}\mathcal{J}(\xi_1-\xi_2) + o(|\xi_1-\xi_2|_X), \text{ as } \xi_1 \to \xi_2.$$

This implies that $u(\cdot;\xi,\lambda)$ is differentiable in ξ and its derivative satisfies

$$\begin{aligned} D_\xi u(t;\xi,\lambda) = T(t,\tau)P(\tau) + &\int_\tau^t T(t,s)P(s)D_uF(s,u(s;\xi,\lambda),\lambda)D_\xi u(s;\xi,\lambda)ds \\ &\int_{-\infty}^t T(t,s)Q(s)D_uF(s,u(s;\xi,\lambda),\lambda)D_\xi u(s;\xi,\lambda)ds. \end{aligned}$$

Now we prove that $|I|_{E_\tau^-(\gamma-\delta,X)} = o(|\xi_1-\xi_2|)$ as $\xi_1 \to \xi_2$. Let

$$\begin{aligned} I_1 =& e^{(\gamma-\delta)t}\{|\int_N^t T(t,s)P(s)[F(s,u(s;\xi_1,\lambda),\lambda) - F(s,u(s;\xi_2,\lambda),\lambda) \\ &- D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]ds|_X\} \end{aligned}$$

for $t \le N < \tau$ and $I_1 = 0$ for $t > N$;

$$\begin{aligned} I_2 =& e^{(\gamma-\delta)t}\{|\int_\tau^N T(t,s)P(s)[F(s,u(s;\xi_1,\lambda),\lambda) - F(s,u(s;\xi_2,\lambda),\lambda) \\ &- D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]ds|_X\}; \\ I_3 =& e^{(\gamma-\delta)t}\{|\int_N^\tau T(t,s)Q(s)[F(s,u(s;\xi_1,\lambda),\lambda) - F(s,u(s;\xi_2,\lambda),\lambda) \\ &- D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s\xi_2,\lambda))]ds|_X\}; \\ I_4 =& e^{(\gamma-\delta)t}\{|\int_{-\infty}^N T(t,s)P(s)[F(s,u(s;\xi_1,\lambda),\lambda) - F(s,u(s;\xi_2,\lambda),\lambda) \\ &- D_uF(s,u(s;\xi_2,\lambda),\lambda)(u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda))]ds|_X\}; \end{aligned}$$

where $-N$ is a large number to be chosen later.

It is sufficient to show that for any $\varepsilon > 0$ there is a $\sigma > 0$ such that if $|\xi_1 - \xi_2| \le \sigma$, then $|I|_{E_\tau^-(\gamma-\delta,X)} \le \varepsilon$. A simple computation implies that

$$\begin{aligned} I_1 &\le 2Lip_uFM_2 \int_t^N e^{(\gamma-\delta)t-\alpha(t-s)}|u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda)|_X ds \\ &\le 2Lip_uFM_2 \int_t^N e^{(\gamma-\delta)t-\alpha(t-s)-(\gamma-2\delta)s}ds|u(\cdot;\xi_1,\lambda)-u(\cdot;\xi_1,\lambda)|_{E_\tau^-(\gamma-2\delta,X)} \\ &\le \frac{2Lip_uFM_1M_2e^{\gamma\tau}}{1-K(\beta-\gamma+2\delta,\gamma-2\delta-\alpha,\rho)Lip_uF}(\gamma-\delta-\alpha)^{-1}e^{\delta N}|\xi_1-\xi_2|_X. \end{aligned}$$

Choose $-N$ so large that

$$\frac{2Lip_u F M_1 M_2 e^{\gamma\tau}}{1-K(\beta-\gamma+2\delta,\gamma-2\delta-\alpha,\rho)Lip_u F}(\gamma-\delta-\alpha)^{-1}e^{\delta N}\leq\frac{1}{4}\varepsilon.$$

Hence for such N we have that

$$\sup_{t\leq\tau} I_1\leq\frac{1}{4}\varepsilon|\xi_1-\xi_2|_X.$$

Fix such N, for I_2 we have that

$$\begin{aligned}I_2 &\leq M_2\int_\tau^N e^{(\gamma-\delta)t-\alpha(t-s)}\int_0^1 |D_uF(s,\theta u(s;\xi_1,\lambda)+(1-\theta)u(s;\xi_2,\lambda),\lambda)\\ &\qquad -D_uF(s,u(s;\xi_2,\lambda),\lambda)|_Y d\theta|u(s;\xi_1,\lambda)-u(s;\xi_2,\lambda)|_X ds\\ &\leq\frac{M_1M_2e^{\gamma\tau}}{1-K(\beta-\gamma+\delta,\gamma-\delta-\alpha,\rho)Lip_uF}\int_\tau^N e^{(\gamma-\delta-\alpha)(\tau-s)}\\ &\int_0^1|D_uF(s,\theta u(s;\xi_1,\lambda)+(1-\theta)u(s;\xi_2,\lambda),\lambda)-D_uF(s,u(s;\xi_2,\lambda),\lambda)|_Y d\theta ds|\xi_1-\xi_2|_X.\end{aligned}$$

The last integral is on the compact interval $[N,\tau]$. By the continuity of $u(s;\xi,\lambda)$, we have that there is a $\sigma_1>0$ such that if $|\xi_1-\xi_2|\leq\sigma_1$, then

$$\sup_{t\leq\tau} I_2\leq\frac{1}{4}\varepsilon|\xi_1-\xi_2|_X.$$

Therefore if $|\xi_1-\xi_2|\leq\sigma_1$, then

$$\sup_{t\leq\tau} I_1+\sup_{t\leq\tau} I_2\leq\frac{1}{2}\varepsilon|\xi_1-\xi_2|_X.$$

Similarly, there exists $\sigma_2>0$ such that if $|\xi_1-\xi_2|\leq\sigma_2$, then

$$\sup_{t\leq\tau} I_3+\sup_{t\leq\tau} I_4\leq\frac{1}{2}\varepsilon|\xi_1-\xi_2|_X.$$

Taking$\sigma=min\{\sigma_1,\sigma_2\}$, we have that if $|\xi_1-\xi_2|_X\leq\sigma$, then

$$|I|_{E_\tau^-(\gamma-\delta,X)}\leq\varepsilon|\xi_1-\xi_2|_X.$$

Therefore $|I|_{E_\tau^-(\gamma-\delta,X)}=o(|\xi_1-\xi_2|_X)$ as $\xi_1\to\xi_2$. We now prove that $D_\xi u(\cdot;\cdot,\lambda)$ is continuous from X to $E_\tau^-(\gamma,X)$. For $\xi_1,\xi_2\in X$ let

$$\begin{aligned}\bar{I}&=\int_\tau^t T(t,s)P(s)[D_uF(s,u(s;\xi_1,\lambda),\lambda)-D_uF(s,u(s;\xi_2,\lambda),\lambda)]D_\xi u(s;\xi_2,\lambda))]ds\\ &\quad+\int_{-\infty}^t T(t,s)Q(s)[D_uF(s,u(s;\xi_1,\lambda),\lambda)-D_uF(s,u(s;\xi_2,\lambda),\lambda)]D_\xi u(s;\xi_2,\lambda))]ds.\end{aligned}$$

We claim that $|\bar{I}|_{E_\tau^-(\gamma,X)} = o(1)$ as $\xi_1 \to \xi_2$. Using this claim and (3.16), we have that

$$\begin{aligned}
&|D_\xi u(\cdot;\xi_1,\lambda) - D_\xi u(\cdot;\xi_2,\lambda)|_{E_\tau^-(\gamma,X)} \\
&\leq \frac{1}{1-K(\beta-\gamma,\gamma-\alpha,\rho)Lip_u F} \\
&(M_1 e^{\gamma\tau}|\xi_1-\xi_2|_X + |I|_{E_\tau^-(\gamma,X)}) \to 0 \text{ as, } \xi_1 \to \xi_2.
\end{aligned}$$

Hence $D_\xi u(\cdot;\cdot,\lambda)$ is continuous from X to $E_\tau^-(\gamma,X)$. The proof of this claim is similar to the last claim. We omit it. Using the same arguments, we can show that $u(\cdot;\xi,\cdot)$ is C^1 from Λ to $E_\tau^-(\gamma,X)$. Now we show that u is C^k from $X\times\Lambda$ to $E_\tau^-(k\gamma,X)$ by induction. By the induction assumption, we know that u is C^{k-1} from $X\times\Lambda$ to $E_\tau^-((k-1)\gamma,X)$ and $(k-1)$-derivative is Lipschitz. Let us first look at $D_\xi^{k-1}u(t;\xi,\lambda)$. It satisfies the following equation

$$\begin{aligned}
D_\xi^{k-1}u &= \int_\tau^t T(t,s)P(s)D_uF(s,u,\lambda)D_\xi^{k-1}u ds \\
&+ \int_{-\infty}^t T(t,s)Q(s)D_uF(s,u,\lambda)D_\xi^{k-1}u ds \\
&+ \int_\tau^t T(t,s)P(s)R_{k-1}(s,\xi,\lambda)ds + \int_{-\infty}^t T(t,s)Q(s)R_{k-1}(s,\xi,\lambda)ds,
\end{aligned} \tag{3.18}$$

where

$$R_{k-1}(s,\xi,\lambda) = \sum_{i=0}^{k-3}\binom{k-2}{i} D_\xi^{k-2-i}(D_uF(s,u(s;\xi,\lambda),\lambda))D_\xi^{i+1}u(s;\xi,\lambda).$$

We note that $D_\xi^i u \in E_\tau^-(i\gamma,X)$ for $i=1,\cdots,k-1$. A simple computation implies that $R_{k-1}(\cdot,\xi,\lambda)\in L^{k-1}(X,E_\tau^-((k-1)\gamma,X))$ and is C^1 in ξ. In order to insure that the above integrals are well-defined one has to require that $\alpha < (k-1)\gamma < \beta$. This is the direct reason why we need the gap condition. By the assumption of Theorem 3.3, we have that $K(\beta-k\gamma,k\gamma-\alpha,\rho)Lip_uF < 1$. Using this fact and the same argument which we used in the case $k=1$, we can show that $D_\xi^{k-1}u(\cdot;\cdot,\lambda)$ is C^1 from X to $L^k(X,E_\tau^-(k\gamma,X))$. Similarly, we can show that u is C^k from $X\times\Lambda$ to $E_\tau^-(k\gamma,X)$. This completes the proof. □

In the following we will present a theorem which will be used to get the stable foliations for evolution equations in the next section. We assume that the nonlinear F satisfies the Hypothesis A and the evolution operator has a pseudo-dichotomy with $J=[\tau,\infty)$. In addition, we assume that there exist constants $\omega, M_5 > 0$ and $M_6 > 0$ such that

$$|T(t,s)x|_X \leq M_5 e^{\omega(t-s)}|x|_X, \text{ for } t\geq s \tag{3.19}$$

$$|T(t,s)y|_X \leq M_6(t-s)^{-\rho}e^{\omega(t-s)}|y|_Y, \text{ for } t > s. \tag{3.20}$$

Fix λ and let $v(t, \eta, \lambda)$ be the solution of the following integral equation

$$v = T(t,\tau)\eta + \int_\tau^t T(t,s)F(s,v,\lambda)ds$$

The condition on F implies that $v(t, \eta, \lambda)$ is C^k in η. By using Lemma 7.1.1 (Henry[20]), we have that

$$|D^i_\eta v(t,\eta,\lambda)|_{L^i(X,X)} \le C(i,\rho,F,M_5,M_6)e^{(i\omega+(2i-1)\mu)(t-\tau)}, \quad \text{for } i = 1,\cdots,k, \tag{3.21}$$

where $C(i, \rho, F, M_5, M_6)$ is a positive constant,

$$\mu = (M_6 Lip_u F \Gamma(1-\rho))^{\frac{1}{1-\rho}},$$

and $\Gamma(s)$ is the Gamma function.

Consider the following integral equation for $t \ge \tau$

$$\begin{aligned} u(t) = T(t,\tau)Q(\tau)\xi + \int_\tau^t T(t,s)Q(s)[F(s,u(s)+v(s,\eta,\lambda),\lambda) - F(s,v(s,\eta,\lambda),\lambda)]ds \\ + \int_\infty^t T(t,s)P(s)[F(s,u(s)+v(s,\eta,\lambda),\lambda) - F(s,v(s,\eta,\lambda),\lambda)]ds, \end{aligned} \tag{3.22}$$

where $\xi \in X$ and $u \in E^+_\tau(\gamma, X)$, $\alpha < \gamma < \beta$.

THEOREM 3.4. . *Assume that the evolution operator $T(t,s)$ has a pseudo-dichotomy on Z and F satisfies Hypothesis A with $J = [\tau,\infty)$. If there exist $\gamma > 0$, $\alpha < \gamma \le k\gamma < \beta$, and $0 < r \le k-1$ such that*

$$K(\beta - k\gamma, \gamma - \alpha, \rho) Lip_u F < 1, \tag{3.23}$$

$$\gamma - (r\omega + (2r-1)\mu) > 0, \tag{3.24}$$

then for each (ξ, η, λ) (3.22) has a unique solution $u(\cdot;\xi,\eta,\lambda) \in E^+_\tau(\gamma, X)$ which has the following properties

(i) u is C^k from X to $E^+_\tau(\gamma, X)$ with respect to ξ and $D^i_\xi u$, $i = 1,\cdots,k$, are continuous in all variables.

(ii) u is C^r from X to $E^+_\tau(\gamma, X)$ with respect to η and $D^i_\eta u$, $i = 1,\cdots,r$, are continuous in all variables.

(iii) $D^i_\eta u$ is C^{k-1-i} with respect to ξ, $0 \le i \le r$.

Proof. Let $\mathcal{G}$ be the right hand side of (3.22), i.e.,

$$\begin{aligned} \mathcal{G}(u,\xi,\eta,\lambda) = T(t,\tau)Q(\tau)\xi + \int_\tau^t T(t,s)Q(s)[F(s,u(s)+v(s,\eta,\lambda),\lambda) - F(s,v(s,\eta,\lambda),\lambda)]ds \\ + \int_\infty^t T(t,s)P(s)[F(s,u(s)+v(s,\eta,\lambda),\lambda) - F(s,v(s,\eta,\lambda),\lambda)]ds. \end{aligned}$$

From the definition of the pseudo-dichotomy and the condition on F we have that $\mathcal{G}$ is well-defined from $E_\tau^+(\gamma, X) \times X \times X \times \Lambda$ to $E_\tau^+(\gamma, X)$ and is continuous. And for each $\xi_1, \xi_2 \in X$, we have that

$$|\mathcal{G}(u, \xi_1, \eta, \lambda) - \mathcal{G}(u, \xi_2, \eta, \lambda)|_{E_\tau^+(\gamma, X)} \leq M_3 e^{\gamma\tau} |\xi_1 - \xi_2|_X.$$

For each $u, \bar{u} \in E_\tau^+(\gamma, X)$ we have that

$$\begin{aligned} &|\mathcal{G}(u, \xi, \eta, \lambda) - \mathcal{G}(\bar{u}, \xi, \eta, \lambda)|_{E_\tau^+(\gamma, X)} \\ &\leq K(\beta - \gamma, \gamma - \alpha, \rho) Lip_u F |u - \bar{u}|_{E_\tau^+(\gamma, X)}. \end{aligned}$$

(3.23) implies that $\mathcal{G}$ is a uniform contraction with respect to the parameters ξ, η and λ. By using the unform contraction mapping theorem, we have that $\mathcal{G}$ has a unique fixed point $u(\cdot; \xi, \eta, \lambda) \in E_\tau^+(\gamma, X)$ which is continuous in all variable and is Lipschitz continuous with respect to ξ. Furthermore, we have that

$$|u(\cdot; \xi_1, \eta, \lambda) - u(\cdot; \xi_2, \eta, \lambda)|_{E_\tau^+(\gamma, X)} \leq \frac{M_3 e^{\gamma\tau}}{1 - K(\beta - \gamma, \gamma - \alpha, \rho) Lip_u F} |\xi_1 - \xi_2|_X$$

In other words, u is the unique solution of the equation (3.22).

Now let us look at the smoothness of the solution u. By (3.23), we have that there is a positive number δ such that $k\gamma + k\delta < \beta$ and

$$K(\beta - (k\gamma + k\delta), \gamma - \alpha, \rho) Lip_u F < 1.$$

Using the contraction mapping principle, we have that $u \in E_\tau^+(k\gamma + k\delta, X)$. First let us look at the smoothness of u with respect to ξ. Using the same arguments as we used in Theorem 3.3, we have that u is C^k from X to $E_\tau^+(k\gamma, X)$ in ξ. Next we consider the smoothness of the solution u of (3.22) with respect to η. We claim that $u : X \to E_\tau^+((k-i)\gamma + (k-i)\delta, X)$ is C^i in η, for $i \leq r \leq k-1$. The idea of the proof of this claim is the same as in Theorem 3.3. Instead of giving the details of the proof we point out the following. First let us consider the case $r = 1$. Formally differentiatting u in (3.22) with respect to η, we have that

$$\begin{aligned} D_\eta u(t; \xi, \eta, \lambda) = &\int_\tau^t T(t, s) Q(s) D_u F(s, u + v, \lambda) D_\eta u(s; \xi, \eta, \lambda) ds \\ &+ \int_\infty^t T(t, s) P(s) D_u F(s, u + v, \lambda) D_\eta u(s; \xi, \eta, \lambda) ds \\ &+ \int_\tau^t T(t, s) Q(s) (D_u F(s, u(s; \xi, \eta, \lambda) + v(s, \eta, \lambda), \lambda) \\ &\qquad - D_u F(s, v(s, \eta, \lambda), \lambda)) D_\eta v(s, \eta, \lambda) ds \\ &+ \int_\infty^t T(t, s) P(s) (D_u F(s, u(s; \xi, \eta, \lambda) + v(s, \eta, \lambda), \lambda) \\ &\qquad - D_u F(s, v(s, \eta, \lambda), \lambda)) D_\eta v(s, \eta, \lambda) ds. \end{aligned} \tag{3.25}$$

By (3.21), we have

$$|D_\eta v(t,\eta,\lambda)|_{L(X,X)} \le C(1,\rho,F,M_5,M_6)e^{(\omega+\mu)(t-\tau)}.$$

The condition $\gamma-(\omega+\mu)>0$ implies that the integrals in (3.25) are convergent. On the other hand, by (3.23) and the uniform contraction principle, the equation

$$\begin{aligned}U(t;\xi,\eta,\lambda) &= \int_\tau^t T(t,s)Q(s)D_uF(s,u+v,\lambda)U(s;\xi,\eta,\lambda)ds \\ &+ \int_\infty^t T(t,s)P(s)D_uF(s,u+v,\lambda)U(s;\xi,\eta,\lambda)ds \\ &+ \int_\tau^t T(t,s)Q(s)(D_uF(s,u(s;\xi,\eta,\lambda)+v(s,\eta,\lambda),\lambda) \\ &\qquad - D_uF(s,v(s,\eta,\lambda),\lambda))D_\eta v(s,\eta,\lambda)ds \\ &+ \int_\infty^t T(t,s)P(s)(D_uF(s,u(s;\xi,\eta,\lambda)+v(s,\eta,\lambda),\lambda) \\ &\qquad - D_uF(s,v(s,\eta,\lambda),\lambda))D_\eta v(s,\eta,\lambda)ds.\end{aligned}$$

has a unique solution $U\in L(X,E_\tau^+((k-1)\gamma+(k-1)\delta,X))$ which is the derivative of u. Now we consider $D_\eta^r u$, $r>1$. Formally differentiatting u r-time in (3.22) with respect to η, we have that

$$\begin{aligned}D_\eta^r u(t;\xi,\eta,\lambda) &= \int_\tau^t T(t,s)Q(s)D_uF(s,u+v,\lambda)D_\eta^r u(s;\xi,\eta,\lambda)ds \\ &+ \int_\infty^t T(t,s)P(s)D_uF(s,u+v,\lambda)D_\eta^r u(s;\xi,\eta,\lambda)ds \\ &+ \int_\tau^t T(t,s)Q(s)\bar{R}_r(s,\xi,\eta,\lambda)ds + \int_\infty^t T(t,s)P(s)\bar{R}_r(s,\xi,\eta,\lambda)ds,\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}\bar{R}_r(s,\xi,\eta,\lambda) = &\sum_{j=0}^{r-2}\binom{r-1}{j} D_\eta^{r-1-j}(D_uF(s,u(s;\xi,\eta,\lambda)+v(s,\eta,\lambda),\lambda))D_\eta^{j+1}u(s;\xi,\lambda) \\ &D_\eta^{r-1}[(D_uF(s,u(s;\xi,\eta,\lambda)+v(s,\eta,\lambda),\lambda)-D_uF(s,v(s,\eta,\lambda),\lambda))D_\eta v(s,\eta,\lambda)].\end{aligned}$$

Since $D_\eta^i u\in L^i(X,E_\tau^+((k-i)\gamma+(k-i)\delta,X))$ for $i=1,\cdots,r-1$, and

$$|D_\eta^i v(t,\eta,\lambda)|_{L^i(X,X)} \le C(i,\rho,F,M_5,M_6)e^{(i\omega+(2i-1)\mu)(t-\tau)}, \quad \text{for } i=1,\cdots,k,$$

a simple computation implies that $R_r(.,\xi,\eta,\lambda)\in L^r(X,E_\tau^+((k-r)\gamma+(k-r)\delta,X))$. This implies that the integrals in (3.26) are convergent provided $r\le k-1$. On the other hand, by (2.23), (2.24) and the contraction mapping Theorem, we have that the equation

$$\begin{aligned} U(t;\xi,\eta,\lambda) = &\int_\tau^t T(t,s)Q(s)D_uF(s,u+v,\lambda)U(s;\xi,\eta,\lambda)ds \\ &+ \int_\infty^t T(t,s)P(s)D_uF(s,u+v,\lambda U(s;\xi,\eta,\lambda)ds \\ &+ \int_\tau^t T(t,s)Q(s)\bar{R}_r(s,\xi,\eta,\lambda)ds + \int_\infty^t T(t,s)P(s)\bar{R}_r(s,\xi,\eta,\lambda)ds \end{aligned}$$

has a unique solution $U \in L^r(X, E_\tau^+((k-r)\gamma + (k-r)\delta, X))$ which is the derivative of $D_\eta^{r-1}u$. Similarly we can show (iii). This completes the proof. □

§4 Applications.

In this section, we will discuss some direct consequences of Theorem 3.3 and Theorem 3,4. Many differential equations in infinite dimensional spaces such as parabolic equations, hyperbolic equations and delay equations are equivalent to certain forms of integral equations by using the variation of constants formula. This observation leads to the following.

Let Z and Λ be Banach spaces. Consider the following semilinear evolutionary equation

$$\frac{du}{dt} + Au = F(u,\lambda), \tag{4.1}$$

where $u \in Z$, the parameter $\lambda \in \Lambda$. We assume that the linear operator A satisfies

Hypothesis B. *The operator A is a sectorial operator and the spectrum $\sigma(A)$ of A has the following decomposition*

$$\sigma(A) = \sigma_1(A) \cup \sigma_2(A) \tag{4.2}$$

$$-\omega < \inf_{\nu\in\sigma_1(A)} Re\,\nu \leq \sup_{\nu\in\sigma_1(A)} Re\,\nu < \alpha < \beta < \inf_{\nu\in\sigma_2(A)} Re\,\nu, \tag{4.3}$$

where ω, α and β, $\beta > 0$ are constants and $\sigma_1(A)$ is bounded.

It is known that there is a positive number a such that the fractional powers of $(A+aI)$ are well defined, which we denote by $(A+aI)^\theta$, for all $\theta \in \mathbb{R}$. See Henry [20] for example. The domain of $(A+aI)^\theta$, which we denote by Z^θ, is a Banach space under the graph norm $|.|_\theta$. Furthermore $-A$ is the infinitesimal generator of an analytic semigroup, which we denote by e^{-At}.

The nonlinear term F is assumed to satisfy

HYPOTHESIS C. *There exist nonnegative constants* $\theta_1 \le 1$ *and* $\theta_2 \le 1$, $0 \le \theta_1 - \theta_2 < 1$, *such that* F *is a continuous mapping from* Z^{θ_1} *to* Z^{θ_2}.

We note that many differential equations such as reaction-diffusion equations, Cahn-Hilliard equations, Kuramoto-Sivashinsky equations and Navier-Stokes equations can be written in the form of (4.1).

By using Lemma 1.4.3 in [20] we have the following lemma

LEMMA 4.1. *There are positive constants* M_i, $i = 1, \cdots, 6$ *and projection* P *corresponding to* $\sigma_1(A)$ *such that*

$$|e^{-At}P|_{L(Z^{\theta_1}, Z^{\theta_1})} \le M_1 e^{-\alpha t}, \text{ for } t \le 0, \tag{4.4}$$

$$|e^{-At}P|_{L(Z^{\theta_1}, Z^{\theta_2})} \le M_2 e^{-\alpha t}, \text{ for } t \le 0, \tag{4.5}$$

$$|e^{-At}(I-P)|_{L(Z^{\theta_1}, Z^{\theta_1})} \le M_3 e^{-\beta t}, \text{ for } t \ge 0, \tag{4.6}$$

$$|e^{-At}(I-P)|_{L(Z^{\theta_1}, Z^{\theta_2})} \le M_4 t^{\theta_2 - \theta_1} e^{-\beta t}, \text{ for } t > 0, \tag{4.7}$$

$$|e^{-At}|_{L(Z^{\theta_1}, Z^{\theta_1})} \le M_5 e^{\omega t}, \text{ for } t \ge 0, \tag{4.8}$$

$$|e^{-At}|_{L(Z^{\theta_1}, Z^{\theta_2})} \le M_6 t^{\theta_2 - \theta_1} e^{\omega t}, \text{ for } t > 0. \tag{4.9}$$

As an application of Theorem 3.3, we give the following invariant manifold theorem which generalizes the usual center unstable manifold theorem (see, for example, [7])

THEOREM 4.2. . *Assume that the Hypothesis B and C are satisfied,* $F \in C^k(Z_{\theta_1}, Z_{\theta_2})$ *for some integer* $k \ge 1$ *and there is a* $\gamma > 0$ *such that*

$$\alpha < \gamma \le k\gamma < \beta, \tag{4.10}$$

$$K(\beta - k\gamma, \gamma - \alpha, \theta_1 - \theta_2) Lip_u F < 1, \tag{4.11}$$

where K *is given in Lemma 3.1. Then there exists a unique* C^k γ*-unstable manifold* $\mathcal{W}^u_\gamma$ *for (4.1),*

$$\mathcal{W}^u_\gamma = \{u_0 | u(t, u_0) \text{ exists for } t \le 0, u \in E_0^-(\gamma, Z^{\theta_1})\}, \tag{4.12}$$

which is the graph of a C^k *mapping* $h : PZ^{\theta_1} \times \Lambda \to (I-P)Z^{\theta_1}$.

REMARK. *The gap condition (4.10) always holds if* $\alpha \le 0$, *here the* γ*-unstable manifold becomes unstable manifold or center-unstable manifold. If the condition (4.10) fails, then the examples given in [9] show that the invariant manifolds lose smoothness even if the nonlinearities are analytic.*

Proof. From the definition of γ-unstable manifold we have

$$\mathcal{W}^u_\gamma = \{u_0 | u(t, u_0) \text{ exists for } t \le 0, u \in E_0^-(\gamma, Z^{\theta_1})\},$$

where $u(t, u_0)$ is a solution of (4.1) with the initial data $u(0, u_0) = u_0$. It is clear that $\mathcal{W}^u_\gamma$ is invariant under the flows of (4.1). We want to show that $\mathcal{W}^u_\gamma$ is given by the graph of a C^k function over PZ^{θ_1}. First we claim

Claim. $u^0 \in \mathcal{W}^u_\gamma \iff u(\cdot) \in E_0^-(\gamma, Z^{\theta_1})$ with $u(0) = u^0$ and satisfies

$$
\begin{aligned}
u = e^{-APt}\xi + \int_0^t e^{-AP(t-s)}PF(u,\lambda)ds \\
+ \int_{-\infty}^t e^{-AQ(t-s)}QF(u,\lambda)ds,
\end{aligned}
\tag{4.13}
$$

where $Q = I - P$ and $\xi = Pu_0$.

This claim can be easily verified by using the variation of constants formula.

Let $X = Z^{\theta_1}$, $Y = Z^{\theta_2}$, $T(t,s) = e^{-A(t-s)}$, $\tau = 0$ and $P(t) = P$. Then the integral equation (4.13) has the same form as (3.12). It is easy to see that the conditions of Theorem 4.2 are the same as those in Theorem 3.3 in this case. By using Theorem 3.3, we have that for every $(\xi, \lambda) \in PZ^{\theta_1} \times \Lambda$ the integral equation (4.13) has a unique solution $u(.;\xi,\lambda) \in E_0^-(\gamma, Z^{\theta_1})$ which has the following property:

$$u : X \times \Lambda \to E_0^-(k\gamma, X)$$

is a C^k mapping. Let

$$
\begin{aligned}
h(\xi,\lambda) &= Qu(0;\xi,\lambda) \\
&= \int_{-\infty}^0 e^{-AQ(t-s)}QF(u(s;\xi,\lambda),\lambda)ds.
\end{aligned}
$$

Then $h : PZ^{\theta_1} \times \Lambda \to (I-P)Z^{\theta_1}$ is a C^k mapping and satisfies that

$$|h(\xi_1,\lambda) - h(\xi_2,\lambda)|_{Z^{\theta_1}} \le \frac{M_1 K(\beta-\gamma, \gamma-\alpha, \theta_1-\theta_2)Lip_u F}{1 - K(\beta-\gamma,\gamma-\alpha,\theta_1-\theta_2)Lip_u F}|\xi_1 - \xi_2|_{Z^{\theta_1}}. \tag{4.14}$$

Finally, we have that

$$\mathcal{W}^u_\gamma = \{\xi + h(\xi,\lambda) | \xi \in PZ^{\theta_1}\}.$$

This completes the proof. □

As an application of Theorem 3.4, we will prove a theorem on invariant foliations of space Z^{θ_1} in such a way that the leaves of the foliation are transverse to the invariant manifold $\mathcal{W}^u_\gamma$ from Theorem 4.2. If $F(0,\lambda) = 0$, then the unique leaf that passes through 0 is the stable manifold of (4.1). We will also see that this invariant foliation gives us exponentially attractivity. Hence, if $\mathcal{W}^u_\gamma$ is finite dimensional, then this implies the inertial manifold theorem given in [16].

Let $u(t,u_0)$, $t \geq 0$, be the solution of (4.1) with the initial data u_0. By using Lemma 7.1.1 in [20], we have that

$$|D^i_\eta v(t,\eta)|_{L^i(X,X)} \leq C(i,\rho,F,M_5,M_6)e^{(i\omega+(2i-1)\mu)t}, \quad \text{for } i=1,\cdots,k, \tag{4.15}$$

where $C(i,\rho,F,M_5,M_6)$ is a positive constant,

$$\mu = (M_6 Lip_u F\Gamma(1-\rho))^{\frac{1}{1-\rho}},$$

and $\Gamma(s)$ is the Gamma function.

THEOREM 4.3. . *Assume that all conditions in Theorem 4.2 are satisfied. In addition, we assume that there exist γ, $max\{0,\alpha\} < \gamma < \beta$, and r, $0 < r \leq k-1$ such that*

$$(max\{M_1,M_2\}+1)K(\beta-\gamma,\gamma-\alpha,\theta_1-\theta_2)Lip_u F < 1, \tag{4.16}$$

$$\gamma - (r\omega + (2r-1)\mu) > 0. \tag{4.17}$$

Then there exists an unique invariant foliation of Z^{θ_1} whose leaves are γ-stable. Moreover, each leaf is given by

$$\mathcal{W}^s_\gamma(\xi + h(\xi,\lambda)) = \{\zeta + h(\xi,\lambda) + \phi(\xi,\zeta,\lambda) | \zeta \in QZ^{\theta_1}\},$$

where $\xi \in PZ^{\theta_1}$ (regarded as an index set), $\mathcal{W}^s_\gamma(\xi + h(\xi,\lambda))$ is the leaf that passes through $\xi + h(\xi,\lambda)$ and $\phi : PZ^{\theta_1} \times QZ^{\theta_1} \times \Lambda \to PZ^{\theta_1}$ satisfies the following

(i) $\phi(\xi,\cdot,\lambda) : QZ^{\theta_1} \to PZ^{\theta_1}$ is C^k and $D^k_\zeta\phi$ is continuous.

(ii) $\phi(\cdot,\zeta,\lambda) : PZ^{\theta_1} \to PZ^{\theta_1}$ is C^r and $D^r_\xi\phi$ is continuous.

(iii) $D^i_\xi\phi$ is C^{k-i-1} differentiable in ζ for $i \leq r$.

(iv) $\mathcal{W}^s_\gamma(\xi + h(\xi,\lambda))$ intersects $\mathcal{W}^u_\gamma$ transversely at a unique point.

REMARK. *We note that the condition (4.17) holds for $r = k-1$ if the positive real parts of eigenvalues and the Lipschitz constant of the nonlinearity are small enough.*

Proof. For any $\eta \in Z^{\theta_1}$ let $u(t,\eta)$ be the solution of (4.1) with the initial data $u(0,\eta) = \eta$. We are looking for all solutions $u(t,\bar{\eta})$ of (4.1) which are asymptotically equivalent to $u(t,\eta)$ in the sense that $u(.,\bar{\eta}) - u(.,\eta) \in E^+_0(\gamma,X)$. In other wards, we are looking for

$$\mathcal{W}^s_\gamma(\eta) = \{\bar{\eta} \in Z^{\theta_1} | u(.,\bar{\eta}) - u(.,\eta) \in E^+_0(\gamma,X)\}.$$

Set $w(t) = u(t,\bar{\eta}) - u(t,\eta)$. Then w satisfies the following equation

$$\frac{dw}{dt} + Aw = F(w + u(t,\eta),\lambda) - F(u(t,\eta),\lambda). \tag{4.18}$$

Using the same arguments as in Theorem 4.4, we have that $w(t,w_0), t \geq 0$, is a solution of (4.18), which belongs to $E_0^+(\gamma, Z^{\theta_1})$, if and only if $w(\cdot) \in Z^{\theta_1}$ with $w(0) = w_0 = \bar{\eta} - \eta$ and satisfies the following integral equation

$$
\begin{aligned}
w = e^{-AQt}\zeta + &\int_0^t e^{-AQ(t-s)}Q[F(w + u(s,\eta),\lambda) - F(u(s,\eta),\lambda)]ds \\
+ &\int_\infty^t e^{-AP(t-s)}P[F(w + u(s,\eta),\lambda) - F(u(s,\eta),\lambda)]ds,
\end{aligned}
\tag{4.19}
$$

where $\zeta = Qw(0)$.

Let $X = Z^{\theta_1}, Y = Z^{\theta_2}, T(t,s) = e^{-A(t-s)}, \tau = 0$ and $P(t) = P$. Then the integral equation (4.19) has the same form as (3.22). By Theorem 3.4, we have that for every $(\zeta,\eta,\lambda) \in QZ^{\theta_1} \times Z^{\theta_1} \times \Lambda$ the integral equation (4.19) has a unique solution $w(\cdot;\zeta,\eta,\lambda) \in E_0^+(\gamma, Z^{\theta_1})$ which has the following properties:

(i) w is C^k from QZ^{θ_1} to $E_0^+(\gamma, Z^{\theta_1})$ with respect to ζ and $D_\zeta^i w$, $i = 1,\cdots,k$, are continuous in all variables.

(ii) w is C^r from Z^{θ_1} to $E_0^+(\gamma, Z^{\theta_1})$ with respect to η and $D_\eta^i w$, $i = 1,\cdots,r$, are continuous in all variables.

(iii) $D_\eta^i w$ is C^{k-i-1} in ζ for $i \leq r$. Moreover,let

$$
\psi(\zeta,\eta,\lambda) = P(w(0;\zeta,\eta,\lambda) + u(0,\eta)).
\tag{4.20}
$$

Then $\psi(\zeta,\eta,\lambda)$ satisfies

$$
|\psi(\zeta_1,\eta,\lambda) - \psi(\zeta_2,\eta,\lambda)|_{\theta_1} \leq \frac{M_3K(\beta-\gamma,\gamma-\alpha,\theta_2-\theta_1)Lip_uF}{1-K(\beta-\gamma,\gamma-\alpha,\theta_2-\theta_1)Lip_uF}|\zeta_1-\zeta_2|_{\theta_1}.
\tag{4.21}
$$

Since $\bar{\eta} = w(0;\zeta,\eta,\lambda) + u(0,\eta) = P(w(0;\zeta,\eta,\lambda)+u(0,\eta)) + Q(w(0;\zeta,\eta,\lambda)+u(0,\eta)) = \psi(\zeta,\eta,\lambda)$, we have that

$$
\mathcal{W}_\gamma^s(\eta) = \{\bar{\eta} | \bar{\eta} = \psi(\zeta,\eta,\lambda) + \zeta + Q\eta,\ \zeta \in QZ^{\theta_1}\}.
$$

Furthermore, $u(t, \mathcal{W}_\gamma^s(\eta)) \subset \mathcal{W}_\gamma^s(u(t,\eta))$ for $t \geq 0$. This implies that $\mathcal{W}_\gamma^s(\eta)$ gives an invariant foliation of Z^{θ_1}. We claim that $W_\gamma^s(\eta)$ transversely intersects W_γ^u at a unique point $\xi + h(\xi,\lambda)$ for some $\xi \in PZ^{\theta_1}$. First if $\bar{\eta} \in \mathcal{W}_\gamma^s(\eta) \cap \mathcal{W}_\gamma^u$, then there exist ζ_0 and ξ_0 such that

$$
\bar{\eta} = \psi(\zeta_0,\eta,\lambda) + \zeta_0 + Q\eta = \xi_0 + h(\xi_0,\lambda).
$$

This implies that ξ_0 is a solution of $\xi_0 = \psi(h(\xi_0,\lambda) - Q\eta,\eta,\lambda)$. On the other hand, Let $g(\xi,\eta,\lambda) = \psi(h(\xi,\lambda) - Q\eta,\eta,\lambda)$. By (4.14) and (4.21), the condition (4.16) implies that

$$
Lip_\xi\, g < 1.
$$

Namely, g is a contraction in ξ. By the contraction mapping theorem, we have that $\xi_0 = \psi(h(\xi_0,\lambda)-Q\eta,\eta,\lambda)$ has a unique fixed point ξ_0. This implies that $\mathcal{W}^s_\gamma(\eta)$ transversely intersects $\mathcal{W}^u_\gamma$ at $\xi_0 + h(\xi_0,\lambda)$. It is easy to see that $\mathcal{W}^s_\gamma(\eta) = \mathcal{W}^s_\gamma(\xi_0 + h(\xi_0,\lambda))$. It suffices to consider leaves passing through $\eta = \xi + h(\xi,\lambda)$, indexed by $\xi \in PZ^{\theta_1}$. Let

$$\phi(\xi,\zeta,\lambda) = \psi(\zeta,\xi + h(\xi,\lambda),\lambda)).$$

We have, since $Q\eta = h(\xi,\lambda)$, that

$$\mathcal{W}^s_\gamma(\xi + h(\xi,\lambda)) = \{\phi(\xi,\zeta,\lambda) + \zeta + h(\xi,\lambda) | \zeta \in QZ^{\theta_1}\}.$$

This completes the proof. □

If the condition $\gamma - (r\omega + (2r-1)\mu) > 0$ is not valid for any $r \geq 1$, then we can still get more smoothness of $\phi(\xi,\zeta,\lambda)$ with respect to ξ. In fact, we will see that $\phi(\xi,\zeta,\lambda)$ is Hölder continuous in ξ with a small Hölder exponent.

THEOREM 4.4. *Assume that all conditions in Theorem 4.2 are satisfied. In addition, we assume that*

$$(max\{M_1, M_3\} + 1)K(\beta-\gamma,\gamma-\alpha,\theta_1-\theta_2)Lip_u F < 1. \tag{4.22}$$

Then the mapping $\phi : PZ^{\theta_1} \times QZ^{\theta_1} \times \Lambda \to PZ^{\theta_1}$ satisfies the following

(i) $\phi(\xi,.,\lambda) : QZ^{\theta_1} \to PZ^{\theta_1}$ is C^k and $D^k_\zeta\phi$ is continuous.

(ii) $\phi(.,\zeta,\lambda) : PZ^{\theta_1} \to PZ^{\theta_1}$ is C^ε (Hölder continuous), where $\varepsilon > 0$ is a small number.

Proof. It suffices to show that the solution $w(.;\zeta,\eta,\lambda)$ of (4.19) is ε-Hölder continuous from Z^{θ_1} to $E^+_0(\gamma, Z^{\theta_1})$ in η. Since $K(\beta-\gamma,\gamma-\alpha,\theta_1-\theta_2)Lip_u F < 1$, there is a $\delta > 0$ such that $\gamma + \delta < \beta$ and

$$K(\beta-\gamma-\delta,\gamma+\delta-\alpha,\theta_1-\theta_2)Lip_u F < 1$$

Using the contraction mapping theorem, we have that $w(.;\zeta,\eta,\lambda) \in E^+_0(\gamma+\delta, Z^{\theta_1})$. For each $\eta_0, \eta_1 \in Z^{\theta_1}$ we have that

$$
\begin{aligned}
&|e^{\gamma t}(w(t;\zeta,\eta_1,\lambda) - w(t;\zeta,\eta_0,\lambda)|_{\theta_1} \\
&= |e^{\gamma t}\{\int_0^t e^{-AQ(t-s)}Q[F(w(s;\zeta,\eta_1,\lambda) + u(s,\eta_1),\lambda) - F(u(s,\eta_1),\lambda) \\
&- F(w(s;\zeta,\eta_0,\lambda) + u(s,\eta_0),\lambda) + F(u(s,\eta_0),\lambda)]ds \\
&+ \int_\infty^t e^{-AP(t-s)}P[F(w(s;\zeta,\eta_1,\lambda) + u(s,\eta_1),\lambda) - F(u(s,\eta_1),\lambda) \\
&- F(w(s;\zeta,\eta_0,\lambda) + u(s,\eta_0),\lambda) + F(u(s,\eta_0),\lambda)ds\}|_{\theta_1} \\
&\le M_4 Lip_u F \int_0^t (t-s)^{\theta_2-\theta_1} e^{\gamma t-\beta(t-s)-\gamma s}|w(\cdot;\zeta,\eta_1,\lambda) - w(\cdot;\zeta,\eta_0,\lambda)|_{E_0^+(\gamma,Z^{\theta_1})}ds \\
&+ M_2 Lip_u F \int_t^\infty e^{\gamma t-\alpha(t-s)-\gamma s}|w(\cdot;\zeta,\eta_1,\lambda) - w(\cdot;\zeta,\eta_0,\lambda)|_{E_0^+(\gamma+\delta,Z^{\theta_1})}ds \\
&+ 2M_4 Lip_u F \int_0^t (t-s)^{\theta_2-\theta_1} e^{\gamma t-\beta(t-s)-(\gamma+\delta)s}|w(\cdot;\zeta,\eta_0,\lambda)|^{1-\varepsilon}_{E_0^+(\gamma,Z^{\theta_1})}|u(s,\eta_1) - u(s,\eta_0)|^{\varepsilon}_{\theta_1}ds \\
&+ 2M_2 Lip_u F \int_t^\infty e^{\gamma t-\alpha(t-s)-(\gamma+\delta)s}|w(\cdot;\zeta,\eta_0,\lambda)|^{1-\varepsilon}_{E_0^+(\gamma,Z^{\theta_1})}|u(s,\eta_1) - u(s,\eta_0)|^{\varepsilon}_{\theta_1}ds.
\end{aligned}
$$

Taking $\varepsilon > 0$ such that $\varepsilon(\gamma + \omega + \mu) \le (1-\varepsilon)\delta$ and using (4.16), we have that

$$
\begin{aligned}
&|(w.t;\zeta,\eta_1,\lambda) - w(.;\zeta,\eta_0,\lambda)|_{E_0^+(\gamma,Z^{\theta_1})} \\
&\le \frac{C(1,\rho,F,M_5,M_6)K(\beta-\gamma,\gamma-\alpha,\theta_1-\theta_2)Lip_u F}{1 - K(\beta-\gamma,\gamma-\alpha,\theta_1-\theta_2)Lip_u F}|w(.;\zeta,\eta_0,\lambda)|^{1-\varepsilon}_{E_0^+(\gamma,Z^{\theta_1})}|\eta_1 - \eta_0|^{\varepsilon}_{\theta_1}.
\end{aligned}
$$

This completes this proof. □

REMARK. *One can apply Theorem 3.3 and Theorem 3.4 to get invariant manifolds and invariant foliations around periodic orbits, homoclinic orbits and heterclinic orbits for evolutionary equations.*

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