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SIMPLIFIED EQUATIONS FOR LOW MACH NUMBER COMBUSTION WITH STRONG HEAT RELEASE

ANDREW MAJDA† AND KEVIN LAMB‡

Introduction.

The familiar system of reaction-diffusion equations

$$\begin{aligned} \rho_0 C_p \frac{DT}{Dt} &= \triangle T + q_0 \rho_0 K e^{-A/T} Z \\ \frac{DZ}{Dt} &= (Le)^{-1} \triangle Z - K e^{-A/T} Z \end{aligned} \tag{0.1}$$

describes combustion processes at low Mach numbers provided that the heat release is sufficiently weak ([1]); with weak heat release, the hydrodynamic flow field decouples from the equations in (0.1) to leading order. In (0.1) and elsewhere in this paper, T is the temperature, Z is the mass fraction of reactant, ρ is the density, q_0 is the heat release, K is the reactive rate, A is the activation energy, $(Le)^{-1}$ is the Lewis number, and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{3} v_i(x,t)\frac{\partial}{\partial x_i}$ denotes the convective derivative along the fluid velocity, $\vec{v} = {}^t(v_1, v_2, v_3)$. Thus, under the assumptions of weak heat release at low Mach numbers, the hydrodynamic velocity field $\vec{v}$ is a prescribed incompressible velocity and all the effects of combustion involve the reaction-diffusion equations in (0.1). Despite all of the intuition that can be gained from studying (0.1), the heat release is typically not small in practice since for example a typical temperature rise from 300° K to 1500° K during combustion has a non-dimensional heat release of 5.

In this paper, we present a limiting system of equations which describe combustion processes with strong heat release at low Mach numbers in either confined or unbounded regions ([2], [3]). This limiting system of equations allows for large heat release, substantial temperature and density variation, and substantial interaction with the hydrodynamic flow field including the effects of turbulence. Nevertheless, since the detailed effects of the

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nonlinear acoustic waves have been removed, this zero Mach number limiting system is significantly simpler than the complete system of equations of compressible combustion. In this paper, we also explicitly compute and analyze a number of exact solutions of these equations in simple geometries. These solutions illustrate the effects of confinement, curvature, external piston motion, and vorticity production on both the combustion process and the hydrodynamic flow field (see sections 3 and 4 below). In the limit of infinitely thin flame sheets described in section 2, these equations already have a prominent role in the numerical simulation of turbulent combustion in open and closed vessels ([4], [2], [5], [6]). One of the goals of this paper is to stimulate the interest of workers in theoretical combustion and nonlinear P.D.E.'s in a new class of problems involving reaction-diffusion equations coupled with fluid dynamics.

For simple one-step Arrhenius chemistry in a bounded region, Ω, the system of equations for low Mach number combustion with large heat release is given by the following:

(0.2) (a) **Mean Pressure Equation**

$$\frac{d}{dt}P(t) = \frac{\frac{1}{\varepsilon}\int_\Omega \gamma q_0 K\rho Z e^{-A/T}}{vol(\Omega)}$$

(b) **Reaction-Diffusion Equations**

$$\rho\frac{DT}{Dt} = \frac{\gamma-1}{\gamma}\frac{dP}{dt} + \frac{q_0}{\varepsilon}K\rho Z e^{-A/T} + \varepsilon\triangle T$$

$$\rho\frac{DZ}{Dt} = -\frac{K}{\varepsilon}\rho Z e^{-A/T} + (Le)^{-1}\varepsilon \text{ div } (\rho\nabla Z)$$

(c) **Nonhomogeneous Incompressible Fluid Equations**

$$\rho\frac{Dv}{Dt} = -\nabla p + \varepsilon P_R\triangle v$$

$$\text{div } v = (\gamma P(t))^{-1}\left(-\frac{dP}{dt} + \frac{\gamma}{\varepsilon}q_0 K\rho Z e^{-A/T} + \gamma\varepsilon\triangle T\right)$$

(d) **Boundary Conditions**

$$v|_{\partial\Omega} = 0, \frac{\partial T}{\partial n}|_{\partial\Omega} = \frac{\partial Z}{\partial n}|_{\partial\Omega} = 0, \rho = P(t)/T(x,t).$$

Thus, the equations in (0.2) involve the six unknowns, $P(t), T(x,t), Z(x,t), \vec{v}$ where $P(t)$ is the mean pressure; the quantity, $p(x,t)$, is the deviation from the mean pressure and is a Lagrange multiplier for (0.2) (c) — this function is determined afterwards once $\vec{v}$ is known as for the ordinary incompressible Navier-Stokes equations. The quantity P_R is the Prandtl number, γ is the ideal gas constant, and ε is a parameter (see section 2) which measures the ratio of the flame front thickness to the dimensions of Ω — typically, we have $\varepsilon \ll 1$ in practice. From (0.2) (c) the reaction-diffusion terms act as sources of specific volume for the fluid dynamics and also generate an increased mean pressure due to confinement from (0.2) (a); on the other hand, the pressure changes and the fluid velocities, $\vec{v}$, influence the structure of the reaction-diffusion equations in (0.2) (b). Thus, the hydrodynamics and the

chemistry are both strongly coupled in (0.2); however, the effects of nonlinear acoustics are completely absent in (0.2) — the only sources of divergence in the fluid are due to combustion, the effects of expansion or compression of the gas due to nonlinear acoustics are ignored. Next, we summarize the contents of this paper.

In section 1 we present a systematic derivation of the equations of zero Mach number combustion for a general reacting mixture and obtain the system in (0.2) as an extremely special case (this derivation follows our earlier treatment for simplified chemistry from [2]). In section 2, we assume that the parameter ε measuring the ratio of flame thickness to the diameter of Ω is vanishingly small and extract a formal limiting system from (0.2) as $\varepsilon \downarrow 0$. This limiting system (see (2.17) below) defines a new class of free boundary problems where the flame front is a free boundary strongly interacting with the hydrodynamic flow field. We also discuss some of the empirical and quantitative laws for the propagation of these flame fronts that have been used in practice. In section 3, we study the equations in (0.2) in a single space dimension. After introducing Lagrangian mass coordinates, we observe that the equations in (0.2) (c) decouple from (0.2) (a) and (0.2) (b) in this coordinate system so that in fact suitable modifications of the reaction-diffusion system in (0.1) do describe combustion with large heat release in a single space dimension. We also discuss the effects of one-dimensional piston motion in a confined chamber and explicitly solve the one-dimensional flame sheet model that emerges in the limit as the flame front thickness parameter, ε, tends to zero. In section 4, we present some new exact solutions of the free boundary problem for zero Mach number combustion with infinitely thin flame structure presented in section 2 (see (2.17)). These solutions illustrate the effects of curvature of the flame front, confinement of the gas, and production of vorticity in swirling flames on both the combustion process and hydrodynamic flow field. Finally, in section 5 we make a brief list of accessible open problems in the mathematical theory of the equations for zero Mach number combustion with strong heat release. We mention here that in a series of interesting papers, Embid ([7], [8], [9]) has established the well-posedness and short-time existence of solutions for the equations in (0.2) as well as for related equations. Furthermore, in the case without diffusion, Schochet ([10]) has developed a completely rigorous derivation of the equations in (0.2) as a zero Mach number limit from the equations of compressible inviscid combustion. Thus, there is even a rigorous proof which supports the formal derivation given in section 1.

§1: The equations for zero Mach number combustion for a General Reacting Mixture.

1.1 Preliminaries. Here we will present the derivation of the low Mach number combustion equations for a reacting mixture of ideal gases in a bounded domain. Our point of departure is the general equations describing the flow of a reacting mixture of compressible fluids as are given, for example, in Williams [11]. These equations express the following principles of conservation:

(1.1) (a) **Conservation of mass**

$$\frac{D\rho}{Dt} + \rho \text{ div } v = 0$$

(b) **Conservation of momentum**

$$\rho\frac{Dv}{Dt} + \nabla p = \eta\triangle v + \frac{\eta}{3}\nabla(\text{ div } v)$$

(c) **Conservation of energy**

$$\rho\frac{Dh}{Dt} = \frac{Dp}{Dt} + \frac{\eta}{2}\sum_{i,j=1}^{3}\left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta ij(\text{ div } v)\right]^2$$

$$+ \text{ div } (\kappa\nabla T) + \text{ div } \left(\sum_{i=1}^{M} h_i j_i\right)$$

(d) **Conservation of the i-th chemical species** $(i = 1, \ldots, M)$

$$\rho\frac{DY_i}{Dt} = \text{div } j_i + \Phi_i(\rho, T, Y).$$

Next, we discuss the meaning of, and the assumptions made on, the different terms appearing in these equations. The variable t denotes time and x denotes space (usually $x\varepsilon\mathbb{R}^3$), while $v = (v_1, v_2, v_3)$ is the average fluid flow velocity of the mixture and $\frac{D}{Dt} = \frac{\partial}{\partial_t} + v.\nabla$. The quantity ρ is the density of the mixture, p is the pressure, T is the temperature, h is the specific enthalpy of the mixture, h_i is the specific enthalpy of the i-th chemical species and $Y = (Y_1, \ldots, Y_M)$. It is assumed on physical grounds that ρ, p, T and Y_i are non-negative quantities. Moreover, since Y_i is defined by $Y_i = \rho_i/\rho$ where ρ_i is the mass density of the i-th chemical, it follows that the quantities Y_i satisfy the constraint $\sum_{i=1}^{M} Y_i = 1$. In practice, one usually uses this fact to eliminate one of the equations in (1.1) (d).

We restrict ourselves to consider a mixture of ideal gases, i.e. satisfying:

(1.2) (a) **Equation of state**

$$p = R\rho T\left(\sum_{i=1}^{M}\frac{Y_i}{W_i}\right)$$

(b) **Caloric equation**

$$h = \sum_{i=1}^{M} h_i Y_i \text{ where}$$

$$h_i = c_p^i(T - T^\circ) + h_i^\circ$$

Here R is the universal gas constant, W_i is the molecular weight of the i-th chemical species, h_i° is the heat of formation of the i-th species at the reference temperature T°

and c_p^i is the heat capacity at constant pressure for the i-th chemical species and it has been assumed constant for simplicity. We remark for reference that for an ideal gas with c_p^i and c_v^i, its heat capacity at constant pressure and volume respectively, and $\gamma_i = c_p^i/c_v^i$, its corresponding γ-gas constant, one can write c_p^i and c_v^i in terms of γ_i as:

$$\text{(1.3)} \qquad \begin{aligned} &\text{(a)} \qquad c_p^i = \frac{R}{W_i}\frac{\gamma_i}{\gamma_i - 1} \\ &\text{(a)} \qquad c_v^i = \frac{R}{W_i}\frac{1}{\gamma_i - 1} \end{aligned}$$

Furthermore, one can define c_p, c_v and γ for a mixture of ideal gases by:

$$\text{(1.4)} \qquad \begin{aligned} &\text{(a)} \qquad c_p = \sum_{i=1}^{M} c_p^i Y_i \\ &\text{(b)} \qquad c_v = \sum_{i=1}^{M} c_v^i Y_i \\ &\text{(c)} \qquad \gamma = c_p/c_v = \left(\sum_{i=1}^{M} \frac{\gamma_i}{\gamma_i - 1} Y_i\right)\left(\sum_{i=1}^{M} \frac{1}{\gamma_i - 1} Y_i\right)^{-1} \end{aligned}$$

In general γ is not constant, however if $\gamma_i = \gamma_0$ for $i = 1, \ldots, M$ then one can check that $\gamma \equiv \gamma_0$. With c_p, c_v and γ given by (1.4) and (1.3), one can rewrite (1.2) (a) as

$$\text{(1.5)} \qquad p = \frac{\gamma - 1}{\gamma} \rho c_p T.$$

We recall the assumptions made on the transport mechanisms. the mixture has been considered as an isotropic elastic medium, η is the coefficient of shear viscosity. The coefficient of Bulk viscosity has been assumed to be zero. The conduction of heat is governed by Fourier's law and κ is the coefficient of thermal conductivity. We also explain the assumptions to be made on the species diffusion mechanism. The term $j_i (i = 1, \ldots, M)$ represents the diffusion flux of the i-th chemical species and together they satisfy the constraint $\sum_{i=1}^{M} j_i = 0$. As constitutive hypothesis for the j_i's we adopt Onsager's generalization of Fick's law:

$$\text{(1.6)} \qquad j_i = \sum_{j=1}^{M} \mathcal{D}^{ij} \nabla Y_j, \quad i = 1, \ldots, M.$$

The diffusion coefficients $\mathcal{D}^{ij}\, i, j = 1, \ldots M$ cannot be prescribed in an arbitrary manner. Using the requirement $\sum_{i=1}^{M} j_i = 0$, it follows from (1.6) that:

$$\text{(1.7)} \qquad 0 = \sum_{j=1}^{M} \left(\sum_{i=1}^{M} \mathcal{D}^{ij}\right) \nabla Y_j,$$

Furthermore, since the mass fractions must satisfy $\sum_{i=1}^{M} Y_i = 1$, we obtain from (1.7):

$$0 = \sum_{j=1}^{M-1} \left[\sum_{i=1}^{M} (\mathcal{D}^{ij} - \mathcal{D}^{iM}) \right] \nabla Y_j \tag{1.8}$$

Therefore we require that the diffusion coefficients $\mathcal{D}^{ij}$ satisfy:

$$\sum_{i=1}^{M} \mathcal{D}^{ij} = \sum_{i=1}^{M} \mathcal{D}^{iM} \quad \text{for } j = 1, \ldots, M-1. \tag{1.9}$$

There are at least two special circumstances in which the complex expressions for j_i given in (1.6) admits simplification. For example, if we consider a reactive binary mixture, then (1.6) reduces to the well-known Fick's law $j_1 = \rho \mathcal{D} \nabla Y_1$ and $j_2 = \rho \mathcal{D} \nabla Y_2$. The second important case is given by the dilute mixture approximation, in which one of the chemical species is present in very high quantities relative to all the other species. For instance, this situation arises in many combustion processes taking place in air, in which nitrogen plays the role of an inert gas. Since 78% of the air's composition consists of nitrogen, these combustion processes satisfy the assumption of the dilute mixture theory.

If we assume that the first $M-1$ species are present in scarce quantities relative to the last one, then in the dilute mixture approximation these $M-1$ species diffuse as in the binary mixture situation:

$$\text{(a)} \quad j_i = \rho \mathcal{D}^i \nabla Y_i, \quad i = 1, \ldots, M-1, \tag{1.10}$$

with difference that in this case the diffusion coefficients $\mathcal{D}^i$ don't need to be the same. For j_M we have:

$$\text{(b)} \quad j_M = \sum_{j=1}^{M-1} \rho (\mathcal{D}^M - \mathcal{D}^j) \nabla Y_j + \rho \mathcal{D}^M \nabla Y_M$$

One can easily check that for the j_i's given in (1.10) $\sum_{i=1}^{M} j_i = 0$. We also remark that for binary mixtures the dilute mixture approximation (1.10) reduces to Fick's law. Notice that j_M is still given by a complex expression, but provided that $\sum_{i=1}^{M} Y_i = 1$, we can compute Y_M directly from $Y_1, \ldots, Y_{M-1}$ without resorting to equation (1.1) (d) for Y_M. Actually, if we add up the equations for $Y_1, \ldots, Y_M$ and use the fact to be discussed below, that $\sum_{i=1}^{M} \Phi_i = 0$, we obtain

$$\rho \frac{DY}{Dt} = \text{div } (\mathcal{D}^M \nabla Y), \quad \text{where } Y = \sum_{i=1}^{M} Y_i. \tag{1.11}$$

Since initially we assume that $\sum_{i=1}^{M} Y_i = 1$, by uniqueness of the solution of (1.11) we see that $\sum_{i=1}^{M} Y_i = 1$ at all times. In consequence, we can ignore the equation for Y_M and

set $Y_M = 1 - \sum_{i=1}^{M-1} Y_i$. From now on, we will work within the framework of the dilute mixture theory.

Next, we discuss the assumptions made on the reaction mechanism. The source terms Φ_i represent the rate of production of the i-th chemical species per unit volume and are given from the reaction mechanism for the mixture as follows:

We assume that the chemical changes taking place in the mixture are governed by ℓ chemical reactions given on a molar basis by:

$$\sum_{i=1}^{M} \nu'_{ij}\, \mathcal{M}_i \underset{b}{\overset{f}{\rightleftarrows}} \sum_{i=1}^{M} \nu''_{ij}\, \mathcal{M}_i \quad j = 1, \ldots, \ell \tag{1.12}$$

where $\mathcal{M}_i (i = 1, \ldots, M)$ is the chemical symbol for the i-th species and $\nu'_{ij}, \nu''_{ij} (i = 1, \ldots, M$ and $j = 1, \ldots, \ell)$ are the stochiometric coefficients for the i-th species as a reactant and as a product in the j-th reaction respectively. Since mass must be preserved for each one of the reactions in (1.12) the stochiometric coefficients must satisfy:

$$\sum_{i=1}^{M} (\nu''_{ij} - \nu'_{ij}) W_i = 0 \text{ for } j = 1, \ldots, \ell \tag{1.13}$$

The source term Φ_i is given from (1.12) by

$$\Phi_i = W_i \sum_{j=1}^{\ell} (\nu''_{ij} - \nu'_{ij}) R_j, \tag{1.14}$$

where R_j is the reaction rate for the j-th reaction. From (1.13) follows immediately that the source terms Φ_i satisfy

$$\sum_{i=1}^{M} \Phi_i = 0. \tag{1.15}$$

We consider reaction rates R_j given by the law of mass action

$$R_j = k_j^f \Pi_{i=1}^{M} \left(\rho \frac{Y_i}{W_i} \right)^{\nu'_{ij}} - k_j^b \Pi_{i=1}^{M} \left(\rho \frac{Y_i}{W_i} \right)^{\nu''_{ij}}, j = 1, \ldots, \ell. \tag{1.16}$$

The reaction rate constants k_j^f and k_j^b are given semi-empirically by Arrhenius' law for dilute mixtures

$$\begin{aligned} k_j^f &= B_j^f T^{\alpha_j^f} \exp(-E_j^f / RT) \\ k_j^b &= B_j^b T^{\alpha_j^b} \exp(-E_j^b / RT), \quad j = 1, \ldots, \ell. \end{aligned} \tag{1.17}$$

Here E_j^f, E_j^b represent the activation energy for the forward and backward j-th reaction respectively; B_j^f and B_j^b are the corresponding frequency factors; α_j^f, α_j^b are constants usually fitted by experiment whose value usually ranges between 0 and 1.

For the derivation of the zero Mach number combustion equations it is convenient to rewrite the equations for conservation of mass and energy as equations for the pressure and the temperature rather than equations for the density and the enthalpy. This can be done by differentiating (1.2) and using (1.1) (a), (c) and (d). The resulting equation for the pressure is:

$$
\begin{aligned}
\frac{Dp}{Dt} + \gamma p \text{div } v &= (\gamma - 1)\frac{\eta}{2} \sum_{i,j=1}^{3} \left[\frac{\partial v_i}{\partial x_j} - \frac{2}{3}\delta ij(\text{div } v) \right]^2 \\
&+ (\gamma - 1)\text{div } (\kappa \nabla T) + (\gamma - 1) \sum_{i=1}^{M-1} \rho(c_p^i - c_p^M)\mathcal{D}^i \nabla T . \nabla Y_i \\
&+ \gamma \sum_{i=1}^{M-1} RT \left(\frac{1}{W_i} - \frac{1}{W_M} \right) \text{div } (\rho \mathcal{D}^i \nabla Y_i) \\
&+ \sum_{i=1}^{M-1} \left[\gamma RT \left(\frac{1}{W_i} - \frac{1}{W_M} \right) - (\gamma - 1)(h_i - h_M) \right] \Phi_i,
\end{aligned}
\tag{1.18}
$$

and for the temperature

$$
\begin{aligned}
\rho c_p \frac{DT}{Dt} &= \text{div } (\kappa \nabla T) + \frac{Dp}{Dt} + \frac{\eta}{2} \sum_{i,j=1}^{M} \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}(\text{div } v) \right]^2 \\
&+ \sum_{i=1}^{M-1} (c_p^i - c_p^M)\mathcal{D}^i \nabla T . \nabla Y_i - \sum_{i=1}^{M-1} (h_i - h_M)\Phi_i .
\end{aligned}
\tag{1.19}
$$

Next we write the system (1.1) (b), (d), (1.18), (1.19) in the following non-dimensional form, where the superscript λ is used to emphasize the dependence of the solutions of

(1.20) on the parameter λ to be defined below:

(1.20)

(a) **Non-dimensional equation for the pressure**

$$\frac{Dp^\lambda}{Dt} + \gamma p^\lambda \text{div } v^\lambda = \frac{\gamma-1}{2\lambda^2 Re}\sum_{i,j=1}^{3}\left[\frac{\partial v_j^\lambda}{\partial x_i} - \frac{2}{3}\delta_{ij}(\text{div } v^\lambda)\right]^2 +$$

$$\frac{\gamma-1}{PrRe}\frac{\Gamma}{\Gamma-1}\triangle T^\lambda + \sum_{i=1}^{M-1}\frac{\gamma-1}{PrRe}\frac{\Gamma}{\Gamma-1}\frac{\rho^\lambda}{Le^i}(\tilde{c}_p^i - \tilde{c}_p^M m)\nabla T^\lambda.\nabla Y_i^\lambda +$$

$$+\sum_{i=1}^{M-1}\frac{\gamma T^\lambda}{PrReLe^i}\left(\frac{1}{\tilde{W}_i} - \frac{1}{\tilde{W}_M}\right)\text{div }(\rho^\lambda \nabla Y_i^\lambda) +$$

$$+\sum_{i=1}^{M-1}\left[\gamma T^\lambda\left(\frac{1}{\tilde{W}_i} - \frac{1}{\tilde{W}_M}\right) - \frac{\gamma-1}{\Gamma-1}\Gamma(h_i^\lambda - h_M^\lambda)\right]\Phi_i^\lambda$$

(b) **Non-dimensional equation for the velocity field**

$$\rho^\lambda\frac{Dv^\lambda}{Dt} + \lambda^2\nabla p^\lambda = \frac{1}{Re}\left[\triangle v^\lambda + \frac{1}{3}\nabla(\text{div } v^\lambda)\right]$$

(c) **Non-dimensional equation for the temperature**

$$\rho^\lambda c_p^\lambda\frac{DT^\lambda}{Dt} = \frac{1}{2\lambda^2 Re}\frac{\Gamma-1}{\Gamma}\sum_{i,j=1}^{3}\left[\frac{\partial v_i^\lambda}{\partial x_j} + \frac{\partial v_j^\lambda}{\partial x_i} - \frac{2}{3}\delta_{ij}(\text{div } v^\lambda)\right]^2$$

$$+\frac{1}{PrRe}\triangle T^\lambda + \frac{\Gamma-1}{\Gamma}\frac{Dp^\lambda}{Dt} + \sum_{i=1}^{M-1}\frac{1}{PrReLe^i}\rho^\lambda(\tilde{c}_p^i - \tilde{c}_p^M)\nabla T^\lambda.\nabla Y_i^\lambda$$

$$-\sum_{i=1}^{M-1}(h_i^\lambda - h_M^\lambda)\Phi_i^\lambda.$$

(d) **Non-dimensional equation for the i-th chemical species**

$$\rho^\lambda\frac{DY_i^\lambda}{Dt} = \frac{1}{PrReLe^i}\text{div }(\rho^\lambda\nabla Y_i^\lambda) + \Phi_i^\lambda, \quad i = 1,\ldots,M-1,$$

where $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + v^\lambda.\nabla.$

In the above equations we used the following:

(1.21) **Non-dimensionalization**

(1) The distance is given in units of a typical length L, e.g. $L =$ diam Ω.

(2) The velocity v^λ is given in units of a typical velocity $|v_m|$ for the problems; $|v_m|$ is often chosen as the free space burning velocity.

(3) The unit of time is determined from (1) and (2) by $\tau = L/|v_m|$.

(4) The pressure p^λ is given in units of the essentially constant initial pressure p_0.

(5) The temperature T^λ is given in units of a typical temperature T_m for the problem; T_m is often chosen as the burnt gas temperature.

(6) The mass fractions Y_i are already non-dimensional i.e. $Y_i^\lambda = Y_i$.

(7) $\tilde{c}_p^i = \tilde{c}_p^i / c_p^m$ is the non-dimensional heat capacity for the i-th chemical species.

$\tilde{W}_i = W_i / W_m$ is the non-dimensional molecular weight for the i-th chemical species. Here c_p^m and W_m are typical values of the heat capacity and the molecular weight for the mixture.

(8) The unit of density ρ_m is given by $p_m = \frac{R}{W_m}\rho_m T_m$.

(9) $Re = \frac{\rho_m L|v_m|}{\eta}$ is the Reynolds number. $Pr = \frac{\eta c_p^m}{\kappa}$ is the Prandtl number. $Le^i = \frac{\kappa}{\rho_m c_p^m \mathcal{D}^i}$ is the Lewis number for the i-th chemical species.

(10) $\lambda = \frac{1}{M\sqrt{\Gamma}}$ where $M = \frac{|v_m|}{\sqrt{\frac{\Gamma P_0}{\rho_m}}}$ is the Mach number and $\Gamma > 1$ is given by $\frac{\Gamma - 1}{\Gamma} = \frac{R}{W_m c_p^m}$.

The above non-dimensionalization, together with the equation of state (1.2) (a), implies the relation

$$p^\lambda = \rho^\lambda T^\lambda \left(\sum_{i=1}^{M} \frac{Y_i^\lambda}{\tilde{W}_i} \right) \tag{1.22}$$

Also from (1.2) (b) one gets

$$\begin{aligned} h_i^\lambda &= \frac{h_i}{c_p^m T_m} = \tilde{c}_p^i T^\lambda + \tilde{h}_i^\circ, \text{ where} \\ \tilde{h}_i^\circ &= (-c_p^i T^\circ + h_i^\circ)/c_p^m T_m, \end{aligned} \tag{1.23}$$

and from (1.4)

$$c_p^\lambda = \sum_{i=1}^{M-1} (\tilde{c}_p^i - \tilde{c}_p^M) Y_i^\lambda \tag{1.24}$$

From (1.14), (1.16) and (1.17) one sees that Φ_i^λ is given by

$$\Phi_i^\lambda = \frac{L\Phi_i}{\rho_m |v_m|} = \tilde{W}_i \sum_{j=1}^{\ell} (\nu_{ij}'' - \nu_{ij}') R_j^\lambda, i = 1, \ldots, M \tag{1.25}$$

where R_j^λ is given by

$$\begin{aligned} R_j^\lambda = D_j \Bigg\{ &(\rho^\lambda)^{|\nu_j'|} (T^\lambda)^{\alpha_j^f} \exp(-A_j^f / T^\lambda) \Pi_{k=1}^M \left(\frac{Y_k^\lambda}{\tilde{W}_k} \right)^{\nu_{kj}'} - \\ &- K_j^{-1} (\rho^\lambda)^{|\nu_j''|} (T^\lambda)^{\alpha_j^b} \exp(-A_j^b / T^\lambda) \Pi_{k=1}^M \left(\frac{Y_k^\lambda}{\tilde{W}_k} \right)^{\nu_{kj}''} \Bigg\}. \end{aligned} \tag{1.26}$$

Here $|\nu_j'| = \sum_{i=1}^{M} \nu_{ij}'$, $|\nu_j''| = \sum_{i=1}^{M} \nu_{ij}''$, $A_j^f = E_j^f/RT_m$ and $A_j^b = E_j^b/RT_m$ are the non-dimensional activation energies for the forward and backward j-th reaction respectively. $D_j = \frac{L}{|v_m|\tau_j^f}$ if Damköhler's first similarity group ($j = 1, \ldots, \ell$). $K_j^{-1} = \frac{\tau_j^f}{\tau_j^b}$ is the non-dimensional equilibrium constant for the j-th reaction and τ_j^f, τ_j^b represent the characteristic time for the occurrence of the forward and backward j-th reaction respectively and are given by

$$(\tau_j^f)^{-1} = B_j^f T_m^{\alpha_j^f} \left(\frac{\rho_m}{W_m}\right)^{|\nu_j'|-1}, \text{ and}$$

$$(\tau_j^b)^{-1} = B_j^b T_m^{\alpha_j^b} \left(\frac{\rho_m}{W_m}\right)^{|\nu_j''|-1}, j = 1, \ldots, \ell\ .$$

1.2 Formal Derivation of the equations for zero Mach number combustion. For the derivation of the equations for zero Mach number combustion we will restrict our attention to a bounded domain. The corresponding formulation and derivation for unbounded regions is even simpler and the reader is referred to [4] — the pressure remains constant for unbounded regions. For the case of a viscous reacting gas mixture we supplement equation (1.20) (b) for the velocity field with the no-slip boundary condition

$$v^\lambda|_{\partial\Omega} = 0. \tag{1.27}$$

Also, for simplicity, we impose on T^λ and Y_i^λ the boundary conditions

$$\frac{\partial T^\lambda}{\partial n}|_{\partial\Omega} = 0, \frac{\partial Y_i^\lambda}{\partial n}|_{\partial\Omega} = 0, i = 1, \ldots, M-1 \tag{1.28}$$

guaranteeing the thermal insulation and impermeability of the boundary.

In the derivation given below, use will be made of the following known facts (Temam [12]):

Fact # 1 Every vector field v in $L^2(\Omega)$ admits a unique orthogonal decomposition in terms of a solenoidal vector field w and a potential $\nabla\phi$:

$$\begin{gathered} v = w + \nabla\phi \text{ with} \\ \text{div } w = 0, w.n|_{\partial\Omega} = 0 \text{ and} \\ \triangle\phi = \text{div } v, \frac{\partial\phi}{\partial n}|_{\partial\Omega} = v.n|_{\partial\Omega} \end{gathered} \tag{1.29}$$

We denote by P the orthogonal projection of $L^2(\Omega)$ onto the divergence zero (or solenoidal) vector fields and remark that $Pv = 0$ if and only if $v = \nabla\phi$ for some ϕ .

Fact # 2 The boundary value problem

$$\begin{aligned} &\triangle\phi = f \text{ in } \Omega \\ &\frac{\partial\phi}{\partial n}|_{\partial\Omega} = g \text{ has a solution with } \nabla\phi \end{aligned} \tag{1.30}$$

uniquely determined if and only if $\int_\Omega f dx = \int_{\partial\Omega} g ds$.

The derivation of the equations describing low Mach number combustion rests on the following three assumptions:

(1.31)

(a) The non-dimensional Mach number is small, or equivalently, the parameter λ defined in (1.15) (10) is large.

(b) The initial pressure P_0 is spatially uniform within terms of order λ^{-2}.

(c) The initial temperature, mass fractions and velocity are in **chemical-fluid balance** within terms of order λ^{-1}. this last assumption will be explained in detail later.

With these assumptions in mind we begin with the Ansatz

$$\begin{aligned} p^\lambda &= p^\infty + \lambda^{-1}p_1 + \lambda^{-2}p_2 + 0(\lambda^{-3}) \\ v^\lambda &= v^\infty + 0(\lambda^{-1}) \\ T^\lambda &= T^\infty + 0(\lambda^{-1}) \\ Y_i^\lambda &= Y_i^\infty + 0(\lambda^{-1}) \quad i = 1, \ldots, M-1. \end{aligned} \tag{1.32}$$

We substitute this expansion into the velocity equation (1.20) (b) and equating the powers of order λ^2 and λ we obtain, respectively

$$\begin{aligned} \nabla p^\infty &\equiv 0 \\ \nabla p_1 &\equiv 0. \end{aligned} \tag{1.33}$$

For the power of order zero in λ we first apply the projection operator p and obtain

$$p\left(\rho^\infty \frac{Dv^\infty}{Dt} - \frac{1}{Re}\nabla w^\infty\right) = 0 \tag{1.34}$$

where $v^\infty = w^\infty + \nabla\phi^\infty$ (we used the fact that $P(\triangle\nabla\phi^\infty) = P(\nabla\triangle\phi^\infty) = 0$ to simplify further equation (1.34)), $\frac{D}{Dt} = \frac{\partial}{\partial t} + v^\infty.\nabla$, and ρ^∞ is given in analogy with (1.22) by

$$p^\infty = \rho^\infty T^\infty \left[\sum_{i=1}^{M-1}\left(\frac{1}{\tilde{W}_i} - \frac{1}{\tilde{W}_M}\right)Y_i^\infty\right]. \tag{1.35}$$

From (1.33), (1.34) and fact #1 we conclude that

(1.36)

(a) $p^\infty \equiv p^\infty(t)$

(b) There exists a scalar pressure π^∞ so that $\rho^\infty \frac{Dv^\infty}{Dt} + \nabla\pi^\infty = \frac{1}{Re}\triangle w^\infty$.

Substituting the expansion in the equations for the temperature and mass fractions (1.20) (c), (d) and collecting terms of order zero in λ gives

(1.37)

$$\text{(a)}\quad \rho^\infty c_p^\infty \frac{DT^\infty}{Dt} = \frac{1}{PrRe}\Delta T^\infty + \frac{\Gamma-1}{\Gamma}\frac{dp^\infty}{dt} + \\ + \sum_{i=1}^{M-1} \frac{\rho^\infty}{PrReLe^i}\left(\tilde{c}_p^i - \tilde{c}_p^M\right)\nabla T^\infty.\nabla Y_i^\infty - \sum_{i=1}^{M-1}(h_i^\infty - h_M^\infty)\Phi_i^\infty$$

$$\text{(b)}\quad \rho^\infty \frac{DY_i^\infty}{Dt} = \frac{1}{PrReLe^i}\text{div}\,(\rho^\infty \nabla Y_i^\infty) + \Phi_i^\infty, i = 1,\ldots,M-1$$

where c_p^∞, h_i^∞ and ϕ_i^∞ have analogous formulas as the ones given in (1.23) to (1.26):

(1.38)

$$\text{(a)}\quad c_p^\infty = \sum_{i=1}^{M-1}(\tilde{c}_p^i - \tilde{c}_p^M)Y_i^\infty$$

$$\text{(b)}\quad h_i^\infty = \tilde{c}_p^i T^\infty + \tilde{h}_i^\circ, \quad i = 1,\ldots,M$$

$$\text{(c)}\quad \Phi_i^\infty = \tilde{W}_i \sum_{j=1}^{\ell}(\nu_{ij}'' - v_{ij}')R_j^\infty, \quad i = 1,\ldots,M, \text{ and}$$

$$\text{(d)}\quad R_j^\infty = D_j\left\{(\rho^\infty)^{|\nu_j'|}(T^\infty)^{\alpha_j^f}\exp(-A_j^f/T^\infty)\Pi_{k=1}^M\left(\frac{Y_k^\infty}{\tilde{W}_k}\right)^{\nu_{kj}'} - \right.$$

$$\left. -K_j^{-1}(\rho^\infty)^{|\nu_j''|}(T^\infty)^{\alpha_j^b}\exp(-A_j^b/T^\infty)\Pi_{k=1}^M\left(\frac{Y_k^\infty}{\tilde{W}_k}\right)^{\nu_{kj}''}\right\}.$$

The derivation of the equation for p^∞ is more subtle than the two previous ones. Substitution of the expansion (1.32) into the pressure equation (1.20) (a) gives

(1.39)

$$\frac{dp^\infty}{dt} = -\gamma p^\infty \Delta\phi^\infty + \mathcal{G}$$

where we used the fact that div $v^\infty = \Delta\phi^\infty$. $\mathcal{G}$ is a second order differential operator in T^∞ and Y^∞ given by

(1.40)

$$\mathcal{G}(p^\infty, T^\infty, Y^\infty) = \frac{\gamma-1}{PrRe}\frac{\Gamma}{\Gamma-1}\Delta T^\infty + \frac{\gamma-1}{PrRe}\frac{\Gamma}{\Gamma-1}\sum_{i=1}^{M-1}\frac{\rho^\infty}{Le^i}(\tilde{c}_p^i - \tilde{c}_p^M)\nabla T^\infty.\nabla Y_i^\infty$$

$$+ \sum_{i=1}^{M-1}\frac{\gamma T^\infty}{PrReLe^i}\left(\frac{1}{\tilde{W}_i} - \frac{1}{\tilde{W}_M}\right)\text{div}\,(\rho\nabla Y_i^\infty) +$$

$$+ \sum_{i=1}^{M-1}\left[\gamma T^\infty\left(\frac{1}{\tilde{W}_i} - \frac{1}{\tilde{W}_M}\right) - \frac{\gamma-1}{\Gamma-1}\Gamma(h_i^\infty - h_M^\infty)\right]\Phi_i^\infty.$$

The left-hand side of (1.39) is a function of time alone while the right-hand side involves functions of both space and time. Therefore, there must exist a scalar function $\mathcal{H}(t)$ of time alone so that

$$\text{(1.41)} \qquad \begin{aligned} &\text{(a)} \quad \frac{dp^\infty}{dt} = \mathcal{H}(t) \\ &\text{(b)} \quad -\gamma p^\infty \triangle \phi^\infty + \mathcal{G} = \mathcal{H}(t). \end{aligned}$$

This function $\mathcal{H}(t)$ which in principle is not specified is in fact completely determined by (1.41) through fact # 2 quoted above:

By (1.41) and (1.29) ϕ^∞ satisfies the boundary value problem

$$\text{(1.42)} \qquad \begin{aligned} &\triangle \phi^\infty = (\gamma p^\infty)^{-1}(\mathcal{G} - \mathcal{H}(t)) \text{ in } \Omega \\ &\frac{\partial \phi^\infty}{\partial n}|_{\partial\Omega} = v^\infty.n|_{\partial\Omega} = 0. \end{aligned}$$

Hence, from fact # 2 we know that the right-hand side of the elliptic equation must satisfy

$$\text{(1.43)} \qquad \int_\Omega (\gamma p^\infty)^{-1}(\mathcal{G} - \mathcal{H}(t))dx = 0,$$

therefore $\mathcal{H}(t)$ is given by the non-local quantity

$$\text{(1.44)} \qquad \mathcal{H}(t) = \frac{\int_\Omega \gamma^{-1}\mathcal{G}dx}{\int_\Omega \gamma^{-1}dx}$$

Going back to (1.41) we see that p is spatially constant and its evolution in time is governed by averaged effects in space through $\mathcal{H}(t)$. Also the potential part of the velocity field $\nabla\phi^\infty$ is completely determined by the chemistry of the mixture through (1.41) (b), whereas the solenoidal part w^∞ obeys the non-homogeneous Navier-Stokes equation (1.36) (b).

Regarding the boundary conditions, it is clear that collecting terms of order zero in λ in (1.28) we get the boundary conditions for T^∞ and Y_i^∞

$$\text{(1.45)} \qquad \frac{\partial T^\infty}{\partial n}|_{\partial\Omega} = 0, \frac{\partial Y_i^\infty}{\partial n}|_{\partial\Omega} = 0 \quad i = 1, \ldots, M-1$$

and collecting the terms of order zero in (1.27) gives

$$\text{(1.46)} \qquad v^\infty|_{\partial\Omega} = 0.$$

Using the orthogonal decomposition of v^∞ given in (1.29) in terms of $\nabla\phi^\infty$ and w^∞ and considering the normal and tangential components of the velocity in $\partial\Omega$ we get the boundary conditions for $\nabla\phi^\infty$ and w^∞:

(1.48) **Equations for zero Mach number combustion**

(a) Nonlinear O.D.E. for the mean pressure p^∞

$$\frac{dp^\infty}{dt} = \mathcal{H}(t) = \frac{\int_\Omega \gamma^{-1}\mathcal{G}dx}{\int_\Omega \gamma^{-1}dx}, p^\infty(0) = 1$$

(b) Elliptic equation for the potential part of the velocity field

$$\triangle\phi^\infty = (\gamma p^\infty)^{-1}(\mathcal{G} - \mathcal{H}(t))$$
$$\frac{\partial\phi^\infty}{\partial n}|_{\partial\Omega} = 0$$

(c) Non-homogeneous incompressible Navier-Stokes equation for the solenoidal part of the velocity field

$$\rho^\infty \frac{Dw^\infty}{Dt} + \nabla\pi^\infty = \frac{1}{Re}\triangle w^\infty - \rho^\infty \frac{D\nabla\phi^\infty}{Dt}$$
$$\text{div } w^\infty = 0$$
$$w^\infty.n|_{\partial\Omega} = 0, \quad w^\infty \times n|_{\partial\Omega} = -\nabla\phi^\infty \times n|_{\partial\Omega}$$

(d) Reaction-diffusion equation for T^∞

$$\rho^\infty c_p^\infty \frac{DT^\infty}{Dt} = \frac{1}{PrRe}\triangle T^\infty + \frac{\Gamma - 1}{\Gamma}\mathcal{H}(t) + \sum_{i=1}^{M-1} \frac{\rho^\infty}{PrReLe^i}(\tilde{c}_p^i - \tilde{c}_p^M)\nabla T^\infty.\nabla Y_i^\infty$$
$$- \sum_{i=1}^{M-1}(h_i^\infty - h_M^\infty)\phi_i^\infty,$$
$$\frac{\partial T^\infty}{\partial n}|_{\partial\Omega} = 0$$

(e) Reaction-diffusion equations for $Y_i^\infty, i = 1, \ldots, M-1$

$$\rho^\infty \frac{DY_i^\infty}{Dt} = \frac{1}{PrReLe^i}\text{div } (\rho^\infty \nabla Y_i^\infty) + \Phi_i^\infty$$
$$\frac{\partial Y_i^\infty}{\partial n}|_{\partial\Omega} = 0$$

(f) $\mathcal{G}$ is given by (1.40), $c_p^\infty, h_i^\infty, \Phi_i^\infty$ are given by (1.38), ρ^∞ is given by (1.35), and $\frac{D}{Dt} = \frac{\partial}{\partial t} + v^\infty.\nabla$.

Restrictions on the Initial Data. In order for (1.32) to represent a self-consistent expansion we must require that the initial data satisfy

(1.49)
(a) $p^\lambda(x,0) = 1 + \lambda^{-2}p_0^2(x) + 0(\lambda^{-3})$
(b) $v^\lambda(x,0) = v_0^\infty(x) + 0(\lambda^{-1})$
(c) $T^\lambda(x,0) = T_0^\infty(x) + 0(\lambda^{-1})$
(d) $Y_i^\lambda(x,0) = Y_{i0}^\infty(x) + 0(\lambda^{-1}) \quad i = 1, \ldots M-1.$

From (1.48) (b) there must be a constant $\mathcal{H}_0$ so that initially

$$\text{(e)} \quad \text{div } v_0^\infty = \gamma^{-1}(Y_0^\infty)[\mathcal{S}(1, T_0^\infty, Y_0^\infty) - \mathcal{H}_0].$$

Of course, we must also require that the boundary conditions be satisfied initially

$$\text{(1.50)} \qquad \begin{aligned} &\text{(a)} \quad \frac{\partial T_0^\infty}{\partial n}|_{\partial\Omega} = 0, \quad \frac{\partial Y_{i0}^\infty}{\partial n}|_{\partial\Omega} = 0, \quad i = 1, \ldots, M-1 \\ &\text{(b)} \quad v_0^\infty|_{\partial\Omega} = 0 \end{aligned}$$

Equation (1.49) (e) constitutes the requirement of approximate chemical-fluid balance within terms of order λ^{-1} for the initial data which we referred to in (1.31). This requirement imposes a constraint only on the gradient part $\nabla\phi_0^\infty$ of the velocity field v_0^∞ and leaves freedom for the choice of the solenoidal component w_0^∞, as long as the requirements

$$w_0^\infty.n|_{\partial\Omega} = 0 \text{ and } w_0^\infty \times n|_{\partial\Omega} = -\nabla\phi_0^\infty \times n|_{\partial\Omega} \text{ are satisfied .}$$

1.3 Evaluation of $\mathcal{H}(t)$ and examples with Simple Chemistry. As we pointed out earlier, the evolution in time of $p^\infty(t)$ is governed by the non-local quantity $\mathcal{H}(t)$. It is therefore of interest to look at the form of $\mathcal{H}(t)$ in more detail. After some manipulation with (1.40) we have the formula,

$$\begin{aligned} \mathcal{H}(t) = \Bigg\{ &- \int_\Omega \frac{1}{PrRe} \sum_{i=1}^{M-1} \frac{1}{\gamma} \left(\frac{1}{\tilde{W}_i} \frac{\gamma_i - \gamma}{\gamma_i - 1} - \frac{1}{\tilde{W}_M} \frac{\gamma_M - \gamma}{\gamma_M - 1} \right) \left(\frac{1}{c_p^\infty} + \frac{\rho^\infty}{Le^i} \right) \cdot \\ &\cdot \nabla T^\infty \cdot \nabla Y_i^\infty dx + \int_\Omega \sum_{i=1}^{M-1} \sum_{j=1}^{\ell} \Bigg[\frac{T^\infty}{\gamma} \left(\frac{1}{\tilde{W}_i} \frac{\gamma_i - \gamma}{\gamma_i - 1} - \frac{1}{\tilde{W}_M} \frac{\gamma_M - \gamma}{\gamma_M - 1} \right) - \\ &- \frac{\gamma - 1}{\gamma} \frac{\Gamma}{\Gamma - 1} (\tilde{h}_i^\circ - \tilde{h}_M^\circ) \Bigg] \tilde{W}_i (\nu_{ij}'' - \nu_{ij}') R_j^\infty dx \Bigg\} \cdot \left(\int_\Omega \gamma^{-1} dx \right)^{-1}. \end{aligned} \tag{1.51}$$

Next we consider two examples with simple kinetics and record the form of these equations.

Example 1. Assume we have two ideal gases, unburnt gas $\mathcal{M}_1$ and burnt gas $\mathcal{M}_2$, both having the same molecular weight W_0 and same γ gas constant γ_0. The unburnt gas $\mathcal{M}_1$ undergoes a one-step irreversible reaction into burnt gas $\mathcal{M}_2$

$$\mathcal{M}_1 \longrightarrow \mathcal{M}_2$$

Using the non-dimensionalization given in (1.21) with $W_m = W_0$, $c_p^m = \frac{R}{W_0} \frac{\gamma_0}{\gamma_0 - 1}$ gives $\tilde{W}_1 = \tilde{W}_2 = 1$, $h_1^\infty = T^\infty + \tilde{h}_1^\circ, h_2^\infty = T^\infty + \tilde{H}_2^\circ$ and $\Gamma = \gamma = \gamma_0$. Substituting these values into (1.51) we get

$$\begin{aligned} \mathcal{H}(t) &= \frac{\gamma_0 Q}{vol\Omega} \int_\Omega R^\infty dx \quad \text{where} \\ Q &= h_1^\circ - h_2^\circ \quad \text{and} \\ R^\infty &= \rho^\infty e^{-(A/T^\infty)} Y_1^\infty \end{aligned} \tag{1.52}$$

The equations (1.48) for zero Mach number combustion in this case become

$$
\begin{aligned}
&\text{(a)} \quad \frac{dp^\infty}{dt} = \frac{\gamma_0 Q}{vol\Omega}\int_\Omega \rho^\infty Y_1^\infty \exp(-A/T^\infty)dx \\
&\text{(b)} \quad \triangle\phi^\infty = \frac{1}{p^\infty}\frac{1}{PrRe}\triangle T^\infty + Q\left(R^\infty - \frac{1}{vol\Omega}\int_\Omega R^\infty dx\right) \\
&\qquad \frac{\partial\phi^\infty}{\partial n}|_{\partial\Omega} = 0 \\
&\text{(c)} \quad \rho^\infty\frac{Dw^\infty}{Dt} + \nabla\pi^\infty = \frac{1}{Re}\triangle w^\infty - \rho^\infty\frac{D\nabla\phi^\infty}{Dt} \\
&\qquad \text{div } w^\infty = 0 \\
&\qquad w^\infty\cdot n|_{\partial\Omega} = 0, \quad w^\infty\times n|_{\partial\Omega} = -\nabla\phi^\infty\times n|_{\partial\Omega} \\
&\text{(d)} \quad \rho^\infty\frac{DT^\infty}{Dt} = \frac{1}{PrRe}\triangle T^\infty + \frac{\gamma_0-1}{\gamma_0}\frac{dp^\infty}{dt} + QR^\infty \\
&\qquad \frac{\partial T^\infty}{\partial n}|_{\partial\Omega} = 0 \\
&\text{(e)} \quad \rho^\infty\frac{DY_1^\infty}{Dt} = \frac{1}{PrReLe}\text{div }(\rho^\infty\nabla Y_1^\infty) - R^\infty \\
&\qquad \frac{\partial Y_1^\infty}{\partial n}|_{\partial\Omega} = 0,
\end{aligned}
\tag{1.53}
$$

where ρ^∞ is given by $p^\infty = \rho^\infty T^\infty$.

Going back to the equation for pressure (1.53) (a), we see that if the reaction is exothermic, i.e., $Q > 0$ then $\mathcal{H}(t) > 0$, and the mean pressure p^∞ in the vessel increases in time. This agrees with the intuitive idea that the heat released by the burning of the gas should increase the pressure inside the closed chamber. Also notice that if we did not impose the conditions of no flux of heat and chemical species (1.45), then we could prevent the rise in pressure, for example, by extracting heat through the boundary.

Example 2. Consider two chemical species $\mathcal{M}_1$ and $\mathcal{M}_2$ with the same γ-gas constant, γ_0, but different molecular weights, $W_1 = W$, and $W_2 = 2W$, undergoing a dissociation-recombination reaction

$$2\mathcal{M}_1 \underset{b}{\overset{f}{\rightleftarrows}} \mathcal{M}_2.$$

If we choose $W_m = W$ and $c_p^m = \frac{R}{W}\frac{\gamma_0}{\gamma_0-1}$ in the non-dimensionalization (1.21), then we have $\tilde{W}_1 = 1, \tilde{W}_2 = 2, h_1^\infty = T^\infty + \tilde{h}_1^\circ, h_2^\circ = \frac{T^\infty}{2} + \tilde{h}_2^\circ$ and $\Gamma = \gamma = \gamma_0$. Substituting these values into (1.51) gives

$$
\begin{aligned}
&\mathcal{H}(t) = \frac{\gamma_0 Q}{vol\Omega}\int_\Omega R^\infty dx \quad \text{where} \\
&Q = \tilde{h}_i^\circ - \tilde{h}_2^\circ \quad \text{and } R^\infty \text{ is given by} \\
&R^\infty = D\left\{(\rho^\infty)^2(Y_1^\infty)^2\exp\left(-\frac{A^f}{T^\infty}\right) - K^{-1}\frac{\rho^\infty}{2}(1-Y_1^\infty)\exp\left(-\frac{A^b}{T^\infty}\right)\right\}.
\end{aligned}
\tag{1.54}
$$

The equations for low Mach number combustion are in this case:

$$
\begin{aligned}
&\text{(a)} \quad \frac{dp^\infty}{dt} = \frac{\gamma_0 Q}{vol\Omega}\int_\Omega R^\infty dx \\
&\text{(b)} \quad \triangle\phi^\infty = \frac{1}{p^\infty}\left\{\frac{1}{PrRe}\triangle T^\infty + \frac{1}{2PrReLe}\text{div }(\rho^\infty T^\infty \nabla Y_1^\infty)\right. \\
&\qquad\qquad \left. + Q\left(R^\infty - \frac{1}{vol\Omega}\int_\Omega R^\infty dx\right)\right\}, \\
&\qquad\qquad \frac{\partial\phi^\infty}{\partial n}|_{\partial\Omega} = 0 \\
&\text{(c)} \quad \rho^\infty\frac{Dw^\infty}{Dt} + \nabla\pi^\infty = \frac{1}{Re}\triangle w^\infty - \rho^\infty\frac{D\nabla\phi^\infty}{Dt}, \\
&\qquad\qquad \text{div } w^\infty = 0 \\
&\qquad\qquad w^\infty \cdot n|_{\partial\Omega} = 0, \quad w^\infty \times n|_{\partial\Omega} = -\nabla\phi^\infty \times n|_{\partial\Omega} \\
&\text{(d)} \quad \frac{p^\infty}{T^\infty}\frac{DT^\infty}{Dt} = \frac{1}{PrRe}\triangle T^\infty + \frac{\gamma_0 - 1}{\gamma_0}\frac{dp^\infty}{dt} + \frac{\rho^\infty}{2PrReLe}\nabla T^\infty \cdot \nabla Y_1^\infty + \\
&\qquad\qquad + \left(\frac{T^\infty}{2} + Q\right)R^\infty, \\
&\qquad\qquad \frac{\partial T^\infty}{\partial n}|_{\partial\Omega} = 0 \\
&\text{(e)} \quad \rho^\infty\frac{DY_1^\infty}{Dt} = \frac{1}{PrReLe}\text{div }(\rho^\infty\nabla Y_1^\infty) - R^\infty, \\
&\qquad\qquad \frac{\partial Y_1^\infty}{\partial n}|_{\partial\Omega} = 0,
\end{aligned}
\tag{1.55}
$$

where in this case ρ^∞ is given by $p^\infty = \rho^\infty T^\infty\left(Y_1^\infty + \frac{Y_2^\infty}{2}\right)$.

If we assume that $Q > 0$, i.e., that the reaction is exothermic in the forward direction, we notice from (1.54) that the pressure will rise whenever the reaction proceeds in the forward direction (i.e., when $R^\infty > 0$) and it will decrease otherwise.

§2: Low Mach Number Combustion with Infinitely thin Flame Structure.

Here we discuss further simplification of the general multi-dimensional equations of zero Mach number combustion derived in (1.48) above. We consider simple one-step irreversible Arrhenius chemistry as described in example 1 above (1.53). We introduce a parameter ε which measures the ratio of the flame thickness to the overall integral length scale. The flame thickness is measured by $\ell_T = \mathcal{K}(\rho_b|v_b|c_p)^{-1}$ and the integral length scale is given by diam Ω; thus, $\varepsilon = \ell_T/\text{diam }\Omega$. For many practical problems, ℓ_T varies over scales smaller than millimeters while diam Ω has the dimensions of meters so we have $\varepsilon << 1$. Here we present the formal simplification that occurs for the equations of zero Mach number combustion in the limit, $\varepsilon << 1$. These simplified equations are very

important for numerical simulation of turbulent combustion ([4], [2], [5], [6]) and define an extremely interesting class of free boundary problems which have not yet been discussed with mathematical rigor. In sections 3 and 4 below, we present several exact solutions of these equations in simplified geometries. By identifying the parameter ε explicitly in the nondimensionalization in (1.21) (see [2]) and repeating the derivation of section one for the equations of zero Mach number combustion with simple chemistry from (1.53), we obtain the equations

(2.1)

(a) **Mean Pressure Equation**

$$\frac{dP(t)}{dt} = \frac{\frac{1}{\varepsilon}\int_\Omega \gamma q_0 K\rho Z e^{-A/T}}{vol(\Omega)}$$

(b) **Elliptic Equation**

$$\triangle\psi = (\gamma P(t))^{-1}\left(-\frac{dP}{dt} + \frac{\gamma}{\varepsilon}q_0 K\rho Z e^{-A/T} + \gamma\varepsilon\triangle T\right)$$

(c) **Nonhomogeneous Fluid Equation**

$$\rho\frac{Dw}{Dt} = -\rho\frac{D}{Dt}(\nabla\psi) - \nabla p + \varepsilon P_R\triangle w$$

(d) **Reaction-Diffusion Equations**

$$\rho\frac{DT}{Dt} = \frac{\gamma-1}{\gamma}\frac{dP}{dt} + \frac{q_0}{\varepsilon}K\rho Z e^{-A/T} + \varepsilon\triangle T$$

$$\rho\frac{DZ}{Dt} = -\frac{K}{\varepsilon}\rho Z e^{-A/T} + (Le)^{-1}\varepsilon\,\mathrm{div}\,(\rho\nabla Z)$$

(e) **Boundary Conditions**

$$\frac{\partial T}{\partial n}|_{\partial\Omega} = \frac{\partial Z}{\partial n}|_{\partial\Omega} = 0$$

$$v = w + \nabla\psi, \quad \mathrm{div}\ w = 0$$

$$w\cdot n|_{\partial\Omega} = 0, \quad \frac{\partial\psi}{\partial n}|_{\partial\Omega} = 0$$

$$w\times n|_{\partial\Omega} = -\triangle\psi\times n|_{\partial\Omega}$$

$$\rho = P(t)/T(x,t)$$

Here we study special initial data in chemical fluid balance with the form for mass-fraction and temperature given by

(2.2)

$$Z(x,0) = \begin{cases} 1, & \phi_o(x) < 0 \\ 0, & \phi_o(x) > 0 \end{cases}$$

$$T(x,0) = \begin{cases} T_u \equiv 1 - q_o, & \phi_0(x) < 0 \\ T_b \equiv 1, & \phi_0(x) > 0 \end{cases}.$$

This initial data including the fluid velocity has a jump discontinuity across the surface $\mathcal{S}_o = \{x \in \Omega | \phi_o(x) = 0\}$ and is a stochiometric mixture composed of unburnt gas for

those points $x \in \Omega$ with $\phi_o(x) < 0$ and burnt gas for those points $x \in \Omega$ with $\phi_o(x) > 0$. After we have finished the discussion in this section, the reader can easily verify that all equations derived below remain valid with obvious modifications for general piecewise smooth initial data $(T(x), Z(x))$ that jump across a surface $\mathcal{S}_o$, provided that the non-dimensional adiabatic equation expressing conservation enthalpy across $\mathcal{S}_o$,

$$T_b(x) = T_u(x) + q_o Z(x) \quad \text{for } x \in \mathcal{S}_o \tag{2.3}$$

is valid at all points of $\mathcal{S}_o$.

Following ideas of Landau [13], under the above assumption and with the special initial data for (2.1) in (2.2), it follows that as $\varepsilon \to 0$, formally the reaction rate is confined to a surface so that

$$\frac{1}{\varepsilon} K Z e^{-A/T_0} \to m(x,t)\delta_{\mathcal{S}(t)}, \quad x \in \mathcal{S}(t). \tag{2.4}$$

Here $\mathcal{S}(t)$ is a surface described by $\mathcal{S}(t) = \{x | \phi(x,t) = 0\}$ with $\phi(x,0) = \phi_o(x)$ and $\delta_{\mathcal{S}(t)}$ is the surface Dirac measure concentrated on $\mathcal{S}(t)$. The function $-m(x,t)$ is the mass flux across $\mathcal{S}(t)$ and is determined by the jump conditions for conservation of mass,

$$\rho_b(v_b \cdot n - V) = \rho_u(v_u \cdot n - V) = -m(x,t) \quad \text{for } x \in \mathcal{S}(t), \ V = \phi_t/|\nabla\phi| \tag{2.5}$$

where n is the outward spatial normal to $\mathcal{S}(t)$. We will say much more about the formal derivation of (2.4) at the end of this section. The equation in (2.5) is a consequence of conservation of mass which is valid for solutions of (2.1) in the limit as $\varepsilon \to 0$. As $\varepsilon \to 0$, from (2.4), the mean pressure and reaction-diffusion equations in (2.1) reduce to

$$\begin{aligned}
&\text{(a)} \quad \frac{dP}{dt} = \frac{\gamma q_o \int_{\mathcal{S}(t)} m(x,t) dA}{vol(\Omega)} \\
&\text{(b)} \quad \frac{DT}{Dt} = \frac{\gamma - 1}{\gamma}\frac{dP}{dt}, \qquad \text{for} \quad \phi(x,t) > 0 \quad \text{and} \quad \phi(x,t) < 0 \\
&\text{(c)} \quad \frac{DZ}{Dt} = 0, \qquad\qquad \text{for} \quad \phi(x,t) > 0 \quad \text{and} \quad \phi(x,t) < 0
\end{aligned} \tag{2.6}$$

where (b) and (c) are supplemented by the jump conditions across $\mathcal{S}(t)$ appropriate for data of the form in (2.2) given by

$$\begin{aligned}
&T_b(x,t) - T_u(x,t) = q_0 \quad \text{for } x \in \mathcal{S}(t) \\
&Z_b(x,t) = 0, Z_u(x,t) = 1 \text{ for } x \in \mathcal{S}(t),
\end{aligned} \tag{2.7}$$

while the elliptic equation from (2.1) (b) becomes

$$\begin{aligned}
&\Delta\psi = (\gamma P)^{-1}\left(-\frac{dP}{dt} + \gamma q_o m \delta_{\mathcal{S}(t)}\right) \\
&\frac{\partial\psi}{\partial n}|_{\partial\Omega} = 0.
\end{aligned} \tag{2.8}$$

The nonhomogeneous Navier-Stokes equations from (2.1) (c) reduce to

$$\rho\frac{Dv}{Dt} - \nabla p = 0 \text{ for } \phi(x,t) > 0 \text{ and } \phi(x,t) < 0 \tag{2.9}$$
$$v \cdot n|_{\partial\Omega} = 0$$

with $v = w + \nabla\psi, \text{div } w = 0, w \cdot n|_{\partial\Omega} = 0$. The equations in (2.9) are supplemented by the jump conditions from (2.5) across $\mathcal{S}(t)$ and the density is given in the two regions by

$$\rho = \begin{cases} P(t)/T_u(x,t), & \phi(x,t) < 0 \\ P(t)/T_b(x,t), & \phi(x,t) > 0 \end{cases}. \tag{2.10}$$

The conservation form of the momentum equation in (2.9) implies the familiar Rankine-Hugoniot jump conditions,

$$v_u \times \vec{n} = v_b \times \vec{n} \tag{2.11}$$
$$\rho(V - v \cdot \vec{n})(v_u \cdot \vec{n} - v_b \cdot \vec{n}) = p_u - p_b$$

across the flame front. We make the following important remark:

It is consistent to drop the effects of kinematic viscosity, $\varepsilon P_R \triangle w$ from (2.1) (c) in the derivation sketched above as regards equation (2.9) describing the interior fluid dynamics. However, there is the usual nonuniformity at $\partial\Omega$ caused by the creation of vorticity at the boundary through the no-slip boundary conditions in (2.1) (e). This is the usual difficulty with singular perturbation at boundaries in high Reynolds number flows and is an obvious source of nonuniformity in the above asymptotics. The effects of viscosity through vorticity creation near boundaries are typically retained in the numerical simulations utilizing (2.6), (2.8), and (2.9) ([2], [4], [5], [6]). Here for simplicity in exposition, we simply drop these terms. The main consequence of the assumption made at the beginning of this section is that the flame front is idealized as infinitely thin and represented by the surface $\mathcal{S}(t)$. With the initial data in (2.2), it is very easy to solve the equations in (2.6) (c) to obtain

$$Z(x,t) = \begin{cases} 1, & \phi < 0 \\ 0, & \phi > 0 \end{cases}. \tag{2.12}$$

At this state of the derivation, we have four equations for the four unknowns $P(t), v$, however, an equation for the unknown flame front $\mathcal{S}(t)$ remains to be determined. The jump conditions for conservation of mass from 92.5) determine an equation for the surface $\mathcal{S}(t)$ (described by $\phi(x,t) = 0$) given by

$$\rho_u(v_u \cdot n - \phi_t/|\nabla\phi|) = -m(x,t)$$

or equivalently,

$$\phi_t + \frac{m(x,t)}{\rho_u}|\nabla\phi| + v_u \cdot \nabla\phi = 0 \tag{2.13}$$
$$\phi(x,0) = \phi_o(x).$$

Thus, from (2.13) the points $\vec{r}(t)$ on $\mathcal{S}(t)$ are described by the equation

$$\frac{d\vec{r}}{dt} = \frac{m(x,t)}{\rho_u}\vec{n}(\vec{r}) + v_u(\vec{r}) \tag{2.14}$$

where $\vec{n}(\vec{r})$ is the outward normal to $\mathcal{S}(t)$. Next we indicate how to determine $\rho_u(t)$ from $P(t)$ (similar considerations apply to $\rho_b(t)$). From the equations, for $\phi(x,t) < 0$,

$$\rho_u T_u = P, \quad \rho_u \frac{D}{Dt} T_u = \frac{\gamma - 1}{\gamma}\frac{dP}{dt}$$

it follows that in the unburnt and burnt gas, generally,

$$\frac{D}{Dt}\left(\frac{\rho^\gamma}{P}\right) = 0 \tag{2.15}$$

and therefore, for the special initial data in (2.2)

$$\rho_u(t) = p^{1/\gamma}(t)\rho_u(0) = p^{1/\gamma}(t)\rho_u^o. \tag{2.16}$$

With the above discussion we summarize the *equations for Zero Mach Number Combustion* with *infinitely thin flame structure* for the unknowns $P, \mathcal{S}(t), v$, that we have derived under the main assumption of this section:

(a) **Nonlinear O.D.E. for the Mean Pressure**

$$\frac{dP}{dt} = \frac{q_o\gamma \int_{\mathcal{S}(t)} m(x,t)dA}{vol(\Omega)}$$

(b) **Eikonal Equation for the Flame Front** $\mathcal{S}(t)$

$$\frac{d\vec{r}}{dt} = v_u(\vec{r}) + \frac{m(x,t)}{\rho_u}\vec{n}(\vec{r})$$

(c) **Elliptic Equation**

(2.17)

$$\triangle\phi = (\gamma P)^{-1}\left(-\frac{dP}{dt} + \frac{q_o\gamma m(x,t)\delta_{\mathcal{S}}}{vol(\Omega)}\right)$$

$$\frac{\partial\phi}{\partial n} = 0$$

(d) **Nonhomogeneous Incompressible Euler Equation**

$$\rho\frac{Dw}{Dt} - \nabla\tilde{p}^\infty = -\rho\frac{D\nabla\phi}{Dt}$$

$$\text{div } w = 0$$

$$w \cdot n|_{\partial\Omega} = 0$$

with the orthogonal decomposition $v = \nabla\phi + w$ and the equations $\frac{D}{Dt}(P\rho^{-\gamma}) = 0$ on either side of the flame front.

In the special case when the domain is unbounded and is a channel of the form discussed in [4] the constant pressure approximation $\frac{dP}{dt} = 0$ applies for the system in (2.1) and therefore for the one in (2.17). In this situation, the equations in (2.17) reduce to those introduced by Ghoniem, Chorin, and Oppenheim in [4]. Furthermore, the equations in (2.17) show how these author's formulation can be modified to treat combustion in confined chambers — the solution of the nonlinear O.D.E. in (2.17) (a) can be easily incorporated as a separate fractional step in the numerical algorithm described in [4] (see [2], [5] for numerical simulations).

In the burnt gas region, the temperature is generally nonuniform when Ω is a bounded domain. Once the equations in (2.17) have been solved, $T_b(x,t)$ is determined in $\phi > 0$ by solving the linear boundary value problem for the first order equation

$$\begin{aligned} \frac{D}{Dt}T_b(x,t) &= 0 \\ T_b(x,t)|_{S(t)} &= q_o + \frac{P(t)}{\rho_u(t)} \\ T_b(x,0) &= 1. \end{aligned}$$

The reader can observe that there is still one more unknown in the equations in (2.17), the mass flux $m(x,t)$, which arises in the flame sheet asymptotics described in (2.4). This is our next topic

Empirical and Asymptotic Formulas for the Mass Flux Across the Flame Sheet. As regards empirical flame propagation laws for $m(x,t)$, Landau ([13]) postulated that m should be a functional of ρ_u and P determined by the local laminar flame velocity and others ([14]) have required that m has a functional form determined empirically from experimental data — either turbulent or laminar. A typical form for m is the power law

$$m(\rho_u, P) = c\rho_u^{1-a}P^a \tag{2.18}$$

where $a, c > 0$ are constants with $\frac{1}{2} \leq a < 1$; the case $a = \frac{1}{2}$ corresponds to the laminar flame law (see [14]).

Markstein ([15]) has suggested that the formula in (2.18) for the mass flux suggested by Landau be modified to incorporate the local effects of curvature, i.e.

$$m(x,t) = c_1\rho_u^{1-a}P^1 - c_2\frac{\rho_u}{R_f} \tag{2.19}$$

where R_f is the local radius of curvature of the flame front (in two space dimensions), taken positive if the flame front is convex with respect to the unburnt gas. Clavin and Williams ([16]) and Matalon and Matkowsky ([17]) have derived asymptotic formulas for the mass flux in the flame sheet limit in (2.4) under suitable additional assumptions involving both

a small flame thickness as well as a sufficiently high activation energy. The most general derivation which allows for arbitrarily strong jumps in the hydrodynamics across the flame front is contained in [17] for nearly unity Lewis numbers. This derivation exploits the fact that there is a second length scale, the reaction layer, which is much smaller than the flame thickness length ℓ_T at high activation energies. The result (see (6.1) from [17] for example) of this asymptotic analysis is a correction to the laminar flame velocity which involves both the curvature (as in (2.19)) and the flame stretch. In section 3 and 4 we will present explicit solutions of the zero Mach number combustion equations in (2.17) in special geometries utilizing either the flame laws in (2.18) (see section 3) or also incorporating some of the effects of curvature such as with (2.19) (see section 4). For the special geometries discussed in section 4, we leave it as an exercise for the reader to extend the exact solutions presented there to some of the other flame laws such as those derived in [17]. All of these types of flame laws have been used in a variety of recent numerical simulations with (2.17) ([2], [4], [5], [6]).

§3: Zero Mach Number Combustion in a Single Space Dimension–Piston Driven Flame Sheets.

In this section, we develop several aspects of the equations for zero Mach number combustion given in (2.1) and (2.17) in a single space dimension. First, we emphasize that through the introduction of Lagrangian mass coordinates, the equations in (2.1) in a single space dimension reduce to reaction-diffusion equations coupled with a nonlocal nonlinear O.D.E. for the mean pressure — in particular, for combustion in a single space dimension, a suitable generalization of the reaction diffusion equations in (0.1) is a valid model for combustion with large heat release. We also generalize this construction to the important practical problem involving piston motion in $1-D$. Finally, we take the infinitely thin flame sheet limit for piston driven combustion in $1-D$ following the general procedure sketched above (2.17). Finally we solve the resulting $1-D$ flame sheet equation with piston motion for the turbulent flame laws for the mass flux in (2.18).

The system of equations in (2.17) with one step irreversible Arrhenius kinetics becomes much simpler in a single space variable provided one introduces Lagrangian mass coordinates. We assume that the bounded domain $\overline{\Omega}$ is given by $0 \leq x \leq 1$ and introduce the change of variables,

$$\begin{aligned} q(x,t) &= \int_o^x \rho(s,t)ds \\ t' &= t \end{aligned} \tag{3.17}$$

where $\rho = \frac{P_0(t)}{T_0(x,t)}$. If $M = \int_o^1 \rho$, then the mass coordinate q from (3.1) varies from 0 to M. From our earlier remark, mass is conserved so that

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho v = 0$$

and one computes from this equation and (3.1) the formulae

$$\frac{\partial}{\partial x} = \rho \frac{\partial}{\partial q}, \quad \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \equiv \frac{D}{Dt} = \frac{\partial}{\partial t'}. \tag{3.2}$$

The first observation is that in a single space variable every function is a gradient, so that the momentum equation in (2.22) (b) is trivially satisfied by an appropriate choice of the scalar pressure ∇p and can be ignored. We also observe that it follows from (3.2) that *in a Lagrangian mass coordinate system, the velocity of the fluid does not need to be computed implicitly in the pressure, temperature, and species equations from (2.1) (a), (d), and also that these equations form a closed system.* The velocity is uniquely determined from (2.1) (b), (c) and can be found afterwards (if desired) once the equations in (2.1) (a), (d) are solved in Lagrangian coordinates. Thus, in Lagrangian coordinates, *the equations of zero Mach number combustion in a single space variable become the integro-differential system of reaction diffusion equations*,

$$\begin{aligned}
&\text{(a)} \quad \frac{dP_0}{dt'} = \frac{q_o\gamma}{\varepsilon} K \int_o^M Z_o e^{-A/T_0} dq \\
&\text{(b)} \quad \frac{\partial Z_0}{\partial t'} = (Le)^{-1}\varepsilon \frac{\partial^2}{\partial q^2} Z_o - \frac{K}{\varepsilon} Z_o e^{-A/T_0} \\
&\text{(c)} \quad \frac{\partial T_0}{\partial t'} = \frac{1}{\rho}\frac{\gamma - 1}{\gamma}\frac{dP_0}{dt'} + \frac{q_0 K}{\varepsilon} Z_o e^{-A/T_0} + \varepsilon \frac{\partial}{\partial q}\left(\rho \frac{\partial}{\partial q} T_o\right) \\
&\text{(d)} \quad \rho = \frac{P_0}{T_0}, \frac{\partial T_0}{\partial q} = \frac{\partial Z_0}{\partial q} = 0 \quad \text{at} \quad q = 0, M.
\end{aligned} \tag{3.3}$$

(The double use of q for both heat release and mass coordinate should not cause confusion.) For propagation in all of space or an unbounded half-space, we have the constant pressure approximation and $\frac{dP_0}{dt} \equiv 0$, so that the system in (3.3) becomes a coupled pair of reaction diffusion equations in Lagrangian coordinates. This system has a nonlinear diffusion coefficient in the temperature equation; except for this difference, the equations in (3.3) (b), (c) become the standard system of reaction-diffusion equations which many authors have used as simple model equations for combustion. However, we emphasize here that the previous derivations of reaction-diffusion equations for combustion in $1 - D$ occurred in Eulerian coordinates under a *constant density* approximation which requires small heat release (see [1]). Here, we have derived a similar system under the much weaker assumptions of small Mach number, almost constant pressure, and approximate chemical fluid balance for the initial data; these assumptions allow for arbitrarily strong heat release.

The Zero Mach Number Limit for the Piston Problem in $1 - D$ Reacting Gases. It is well known that the Lagrangian mass coordinates are very useful in studying $1 - D$ piston motions in gas dynamics — we exploit this fact here. We assume that the piston path is given by $X(t)$ with piston velocity $V(t) = \frac{dX}{dt}$ and $X(0) = 1$. Besides the

basic assumptions in the derivation from section 1, we assume additionally that the piston velocity is always order one in the units of $|v_b|$. In this case, it is convenient to develop the low Mach number asymptotic expansion directly in Lagrangian coordinates. Since the total initial mass is M and mass is conserved, q varies over the fixed interval $[0, M]$. We assume the asymptotic expansions

$$\begin{aligned} p(q,t') &= P_o + \gamma \mathcal{M} P_1 + O(\mathcal{M}^2) \\ v(q,t') &= v_o + O(\mathcal{M}) \\ T(q,t') &= v_o + O(\mathcal{M}) \\ Z(q,t') &= Z_o + O(\mathcal{M}) \end{aligned} \tag{3.4}$$

for $0 < q < M$. The important difference in this case from the one dimensional situation described earlier is that the boundary conditions for v become

$$v(0,t') = 0, \quad v(M,t') = V(t). \tag{3.5}$$

We write the $1-D$ equations from section 1 in mass coordinates and substitute the expansions from (3.4). Since every function is a gradient in $1-D$, the momentum equation is trivially satisfied through a scalar pressure. The resulting equations for T_o, Z_o are those given in (3.3) (b). In a fashion we have already discussed in section 1, the order zero equation for the mean pressure yields the compatibility equations

$$\begin{aligned} &\frac{d}{dt'} P_o(t') = \mathcal{H}(t') \\ &\mathcal{H}(t') = -\gamma P_o(t') \rho \frac{\partial v_0}{\partial q} + \varepsilon \gamma \frac{\partial}{\partial q}\left(\rho \frac{\partial T}{\partial q}\right) + \gamma \frac{K}{\varepsilon} q_o Z_o e^{-A/T_0}. \end{aligned} \tag{3.6}$$

where we utilize the parameter ε, defined in section 2. We divide both sides of the second equation by ρ, integrate and use the boundary conditions (especially (3.5)) to get

$$\mathcal{H}(t') = \frac{-\gamma P_o \frac{dX}{dt'}}{X(t')} + \frac{\gamma \frac{K}{\varepsilon} q_o \int_o^M Z_o e^{-A/T_0} dq}{X(t')}. \tag{3.7}$$

Here we have used the fact that

$$\int_o^M \frac{1}{\rho} dq = \int_o^{X(t)} dx = X(t).$$

Thus, the *equations describing zero Mach number combustion in a* $1-D$ *piston chamber are given by*

$$\text{(a)} \quad \frac{dP_0}{dt'} = \frac{-\gamma P_o \frac{dX}{dt'}}{X(t')} + \frac{\gamma \frac{K}{\varepsilon} q_o \int_o^M Z_o e^{-A/T_0} dq}{X(t')} \text{ and} \tag{3.8}$$

(b) the reaction diffusion equations for $T_o(q,t'), Z_o(q,t')$ from (3.3) (b)

as well as the boundary conditions in (3.5). In this case, the velocity is given explicitly by direct integration of the second formula in (3.6), using the formula for $\mathcal{H}(t')$ from (3.7). At zero Mach number, all the geometry of the piston motion is incorporated in changes in the mean pressure $P_o(t')$ and withdrawing (pushing in), the piston is accompanied by a drop (rise) in mean pressure — this is the well-known principle of adiabatic compression.

Low Mach Number Combustion with Infinitely Thin Flame Structure in $1-D$. We examine the Lagrangian zero Mach number equations in a $1-D$ piston chamber and summarized in (3.8), in the limiting case described in section 2 involving infinitely thin flame structure. We show that the equations for nonisobaric flame propagation derived in a different fashion by Carrier, Fendell, and Feldman [18] are the resulting limit system. We also present some exact solution formulae for the explicit flame laws given in (2.18).

We consider special initial data for the equations in (3.8) with the form

$$
\begin{aligned}
T(q,0) &= \begin{cases} T_u = 1 - q_o, & \hat{q} \le q \le M \\ T_b = 1, & 0 \le q \le \hat{q} \end{cases} \\
Z(q,0) &= \begin{cases} 1, & \hat{q} \le q \le M \\ 0, & 0 \le q \le \hat{q} \end{cases}
\end{aligned} \tag{3.9}
$$

Following the same reasoning as described earlier in section 2, we take the formal limit as $\varepsilon \to 0$ for the equations in (3.8). The result is that the boundary between the burnt and unburnt gas, the flame front, is described by a curve $q = \hat{q}(t)$ and the equations from (3.8) become

$$
\begin{aligned}
&\text{(a)} \quad \frac{dP}{dt'} + \frac{\gamma P}{X(t')}\frac{dX}{dt'} = \frac{\gamma q_o m(\hat{q},t')}{X(t')} \\
&\text{(b)} \quad \frac{\partial t}{\partial t'} = \frac{\gamma - 1}{\gamma}\frac{T}{P}\frac{dP}{dt'} + q_o m(\hat{q},t')\delta(q - \hat{q}(t')) \\
&\text{(c)} \quad \frac{\partial Z}{\partial t'} = -m(\hat{q},t')\delta(q - \hat{q}(t')).
\end{aligned} \tag{3.10}
$$

with $m(\hat{q},t')$ the mass flux. Here $\delta(q - \hat{q}(t'))$ is the Dirac measure centered at $\hat{q}(t')$ and the formulae in (b) automatically incorporate the analogous jump conditions as in (2.7). Following (2.13), (2.14), we see from (3.10) (c) that the flame front $\hat{q}(t')$ satisfies the equation

$$
\begin{aligned}
\frac{d\hat{q}}{dt'} &= m(\hat{q},t) \\
\hat{q}(0) &= \hat{q}
\end{aligned} \tag{3.11}
$$

and $Z(q,t')$ is determined by

$$
Z(q,t') = \begin{cases} 1, & q > \hat{q}(t') \\ 0, & q < \hat{q}(t') \end{cases}.
$$

Also, it follows from (2.16) that

$$
\rho_u(t') = p^{1/\gamma}(t')\rho_u^o, \quad T_u(t') = P(t')^{1-\frac{1}{\gamma}}/\rho_u^o. \tag{3.12}
$$

Next we illustrate the use of the infinitely thin flame sheet equations by utilizing the flame law postulate, $m(\hat{q},t') = m(\rho_u(t'),P(t'))$. In this case, we summarize the formulae in (3.9)–(3.12) and obtain that the *equations for zero Mach number combustion in a* $1-D$ *piston chamber with infinitely thin flame structure reduce to the scalar nonlinear O.D.E.* for the mean pressure,

$$\frac{dP}{dt'} + \frac{\gamma P}{X(t')}\frac{dX}{dt'} = \frac{\gamma q_o m(\rho_u(t'),P(t'))}{X(t')}, \tag{3.13}$$

with $\rho_u(t')$ given in (3.12). This is the equation derived in (2.16) of [18] with a change in notation. Once the equation in (3.13) is solved, it is a simple matter to determine the flame location $\hat{q}(t')$ and $T(q,t')$ from successive integration of (3.11) and (3.12). In fact, for the turbulent flame laws in (2.18), it is very easy to integrate (3.13) explicitly and obtain exact solution formulae which can be useful in checking the validity of these model equations through detailed $1-D$ numerical experiments. For these flame laws, we introduce

$$\theta = a + \frac{1-a}{\gamma}, \qquad 0 < \theta < 1$$
$$\hat{Q} = (\gamma-1)q_o Q(\rho_u^o)^{1-a},$$

then the mean pressure $P(t')$ is given explicitly by (with $P(0)=1, X(0)=1$)

$$P(t) = \left((1-\theta)\hat{Q}\int_o^t X^{\gamma-1-\theta\gamma}(s)ds + 1\right)^{\frac{1}{1-\theta}} X^{\frac{-\gamma}{1-\theta}}. \tag{3.14}$$

More general initial data for T, Z than those in (3.9) are easily incorporated in the model, but we omit these straightforward modifications.

§4: Exact Solutions for Special Geometries of the Equations for Zero Mach Number Combustion.

In this section, we present some exact solutions in special geometries of the equations in (2.17) which describe combustion processes in the limit of both low Mach number and for an infinitely thin flame front. These solutions explicitly display the interaction of a variety of effects on the flow field and flame speed, including curvature of the flame front, confinement to a bounded region and the change of vorticity in the wake of swirling flames. They provide interesting simple solutions for code validation and also clearly display the effects of vorticity production at curved flame fronts. These effects have been ignored in numerical modelling to date ([5], [2] and [4]), so these exact solutions provide simple estimates of the effects ignored by these earlier models. It would also be interesting to investigate the nonlinear stability properties of these exact solutions through both numerical and analytic analysis. The cases considered here involve $2-D$ inviscid flows in simple geometries such as planar or circular flame fronts in both bounded and unbounded domains. Cases with the

flame front moving into shear layers are considered. The exact solutions presented below readily generalize for other flame laws such as those in (2.18), (2.19) as well as including flame stretch, etc. — here we include only curvature effects for simplicity.

As discussed in section 2, we need an equation for the mass flux in describing solutions of the equation in (2.17).

We will use the form used by Ghoniem and Knio [5] which is based on some empirical results of Metghalchi and Keck [19]. This form of the mass flux incorporates some of the effects of the curvature of the flame front. It is given by

$$m(x,t) = N_1 \rho_u^{1-\alpha} P^{\alpha+\beta} - N_2 \frac{\rho_u}{R_f}. \tag{I.1}$$

where N_1, N_2, α and β are constants (see (2.19). R_f is the radius of curvature of the flame front, taken positive if the flame front is convex w.r.t. the unburnt gas. N_2 is small and we require R_f to be large enough so that $m > 0$. In [19] α and β were determined by curve fitting some experimental results for the burning of propane at different equivalence ratios. For a more complete discussion see [5] and [19]. Three sets of data given in [19] are

$$(\alpha, \beta) = \begin{cases} (2.27, -0.23) \\ (2.13, -0.17) \\ (2.02, -0.17) \end{cases}. \tag{I.2}$$

In lieu of equation (2.16) we will rewrite the form of the mass flux as

$$m(x,t) = Q_1 P^{1+\eta}(t) - Q_2 \frac{P^{\frac{1}{\gamma}}(t)}{R_f} \tag{I.3}$$

where

$$\eta = \beta + (\alpha - 1)\frac{\gamma - 1}{\gamma}. \tag{I.4}$$

For the data given by Metghalchi and Keck $\eta + 1 > 0$ for any value of $\gamma > 1$ and $\eta > 0$ if $\gamma > 1.22, 1.18$ or 1.20 for the three sets of data respectively. We will use the mass flux from (I.1) with the assumption that $\eta > 0$ and that R_f is sufficiently large so that the mass flux is positive.

Because of the simple geometries involved in the problems considered here we will make some simplifications in the equations derived in (2.17). In all the cases we will consider below the curvature of the flame front will be a function of time only and hence the mass flux is also a function of time only.

The value of Ψ and $\vec{w}$ are not really of interest. What we are interested in is the velocity field $\vec{v}$. We replace the hydrodynamic equations and boundary conditions in (2.17) with the equivalent equations,

$$\vec{\nabla} \cdot \vec{v} = \frac{-1}{\gamma P(t)} \frac{dP}{dt} + \frac{q_0 m(t)}{P(t)} \delta_S$$

$$\vec{v} \cdot \vec{n}|_{\partial\Omega} = 0 \tag{I.5}$$

and

$$\rho \frac{D\vec{v}}{Dt} = \vec{\nabla} \tilde{p}^{\infty}. \tag{I.6}$$

We now turn to a discussion of the jump conditions. The pressure $P(t)$ is continuous across the flame front and the jump conditions for the density and temperature are given in section 2. The Rankine-Hugoniot jump relations for the hydrodynamics are the familiar equations

$$\rho_b(S_n - \vec{v}_b \cdot \vec{n}) = \rho_u(S_n - \vec{v}_u \cdot \vec{n}) \tag{I.7}$$

$$\vec{v}_u \times \vec{n} = \vec{v}_b \times \vec{n} \tag{I.8}$$

and

$$\rho(S_n - \vec{v} \cdot \vec{n})(\vec{v}_u \cdot \vec{n} - \vec{v}_b \cdot \vec{n}) = \tilde{p}_u - \tilde{p}_b \tag{I.9}$$

where

$$S_n = \frac{d\vec{r}}{dt} \cdot \vec{n} \tag{I.10}$$

is the normal velocity of the flame front. As described in section 2, the mass flux m is equal to both the terms in equation (I.7).

Summary of Equations. Since the assumed mass flux, for the simple geometries to be considered here, is a function of time only,

$$m(t) = Q_1 P^{1+\eta}(t) - Q_2 \frac{P^{\frac{1}{\gamma}}(t)}{R_f(t)}, \tag{S.1}$$

the pressure equation can be written as

$$\frac{dP}{dt} = \frac{q_0 \gamma m(t) A(t)}{vol(\Omega)}. \tag{S.2}$$

where $A(t)$ is the surface area of the flame front. In the interior of the regions of burnt and unburnt gases the velocity is determined by the divergence equation

$$\vec{\nabla} \cdot \vec{v} = \frac{-1}{\gamma P(t)} \frac{dP}{dt}, \tag{S.3}$$

the momentum equation

$$\rho \frac{D\vec{v}}{Dt} = \vec{\nabla} \tilde{p}^{\infty}, \tag{S.4}$$

and, in the case of an enclosed region, the boundary condition

$$\vec{v} \cdot \vec{n}|_{\partial\Omega} = 0. \tag{S.5}$$

along with the jump conditions

$$\rho_b(S_n - \vec{v}_b \cdot \vec{n}) = \rho_u(S_n - \vec{v}_u \cdot \vec{n}) = m(t), \tag{S.6}$$

where $S_n = \frac{d\vec{r}}{dt} \cdot \vec{n}$, and

$$\vec{v}_u \times \vec{n} = \vec{v}_b \times \vec{n} \tag{S.7}$$

on $S(t)$. The initial conditions for the velocity field will depend on the problem, as will the far field conditions in the case of an unbounded domain.

The pressure, density and temperature of the gas are related by (see section 2)

$$P = \rho T, \tag{S.8}$$

$$\frac{D}{Dt}(\rho^{\gamma}/P) = 0, \tag{S.9}$$

so that

$$\rho_u(t) = \rho_0 P^{1/\gamma}(t) \tag{S.10}$$

The quantities ρ_b, T_b are determined across the flame front, $S(t)$, by

$$\rho_b = \frac{\rho_u P}{P + q_0 \rho_u} \tag{S.11}$$

and

$$T_b = T_u + q_0 \tag{S.12}$$

From the equation in (S.6) for conservation of mass we obtain the equation

$$S_n = \vec{v}_u \cdot \vec{n} + \frac{m}{\rho_u} \tag{S.13}$$

which governs the position of the flame front. The initial conditions we will use for the density, temperature and pressure are

$$\rho_b(0) = T_b(0) = P(0) = 1 \tag{S.14}$$

$$\rho_u(0) = \rho_o = \frac{1}{1 - q_0}. \tag{S.15}$$

In the following we will not mention the temperature, as it is easily determined from the pressure and density via equation (S.8).

Planar Flame Fronts. *CASE I: Unbounded domain without shear* For the first case we consider a very simple planar problem. The flow depends only on x and t and only the velocity component in the x direction, u, is nonzero. The position of the flame front is given by $x = S(t)$ with the unburnt gas in the region $x > S(t)$.

The right hand side of the pressure equation (S.2) is zero so we immediately see that the pressure is constant and, from equation (S.10), so is the density of the unburnt gas. It follows that the density of the burnt gas at the flame front is equal to one everywhere and so from equation (S.9) we deduce that ρ_b is equal to one away from the flame front as well. Thus

$$\begin{aligned} P(t) &= P(0) = 1 \\ \rho_u(t) &= \rho_0 P^{1/\gamma}(t) \\ \rho_b(t) &= \frac{\rho_0}{1 + q_0 \rho_0} = 1 \end{aligned} \tag{P.1}$$

The divergence and momentum equations for the velocity, equations (S.3) and (S.4), together with the added condition that the flow be uniform as $x \to \pm\infty$ say that u_u and u_b are constant. It follows then, that we are free to choose our reference frame so that

$$u_u = 0. \tag{P.2}$$

Since the mass flux, via (S.1), is constant, say

$$m(t) = Q_1 = m_0, \tag{P.3}$$

we easily obtain from (S.6) that

$$S(t) = \frac{m_0}{\rho_0} t + S_0 \tag{P.4}$$

and

$$u_b = -q_0 m_0. \tag{P.5}$$

CASE II: Unbounded domain with shear We now reconsider the previous case by letting the planar flame front move into a shear layer. As before we let u be the velocity in the x direction, which is the direction the flame front moves in. Let v be the velocity in the y direction where $x - y$ is the usual cartesian coordinate system. Initially the shear is given by

$$v(x, 0) = v_0(x). \tag{P.6}$$

As in the previous case the pressure and densities are constant and are given by (1). Since there is no y dependence the curvature of the flame front remains zero and the mass flux is again constant, given by (3). From the divergence equation (S.3) we have

$$\frac{\partial u}{\partial x} = 0 \tag{P.7}$$

and hence u is constant. Choosing the same reference frame as before we see that the velocities and the position of the flame front are unchanged from the case without shear. Continuity of the tangential velocity gives

$$v_u = v_b \quad \text{at } x = S(t) \tag{P.8}$$

and the y component of the momentum equation gives

$$\frac{\partial v_b}{\partial t} - q_0 m_0 \frac{\partial v_b}{\partial x} = 0. \tag{P.9}$$

A simple calculation then shows that

$$\begin{aligned} v_u(x, t) &= v_0(x) \\ v_b(x, t) &= v_0\left(\frac{x + q_0 m_0 t}{1 + q_0 \rho_0}\right) \end{aligned} \tag{P.10}$$

so the velocity is no longer steady in the burnt gases. The vorticity jump across the flame front is

$$\delta\zeta = \zeta_u - \zeta_b = \left(\frac{\partial v_u}{\partial x} - \frac{\partial v_b}{\partial x}\right)\Bigg|_{S(t)} = q_0 \frac{\partial v_o}{\partial x}\Bigg|_{S(t)}. \tag{P.11}$$

thus the expansion of the burnt gas decrease the vorticity in the shear layer.

CASE III: Bounded region with no shear Next we investigate the effects of containment of the flow in a closed vessel. One immediate change is that the pressure and the density of the unburnt gas are not functions of time. As a consequence the density (and hence the temperature) of the burnt gas will vary not only with time but spatially as well.

As the simplest case we take the region Ω to be a closed cylinder with cross-sectional area A and length L. The axis of symmetry is aligned with the x-axis with the left end at $x = 0$. For this first example in a bounded region we take the flow to be one dimensional. This means that all variables depend only on x and t and that only the velocity component along the axis of symmetry, u, is nonzero. The unburnt gas is to the right of the planar flame front so that the unit normal $\vec{n}$ points in the positive x direction. The pressure is easily determined. Equation (S.2) gives

$$\frac{dP}{dt} = \frac{q_o \gamma}{L} m(t) = BP^{1+\eta}(t) \tag{P.12}$$

where

$$B = \frac{q_0 \gamma}{L} Q_1 > 0. \tag{P.13}$$

With the initial condition (S.14) this can be solved to give

$$P(t) = (1 - \eta B t)^{-\frac{1}{\eta}}. \tag{P.14}$$

The divergence equation gives

$$\frac{\partial u}{\partial x} = -\frac{1}{\gamma P(t)} \frac{dP}{dt} \tag{P.15}$$

which is easily integrated to yield

$$u(x,t) = -\frac{x}{\gamma P(t)} \frac{dP}{dt} + a(t).$$

By using the boundary conditions that $u = 0$ at $x = 0$ and at $x = L$ we have

$$u_b = -\frac{x}{\gamma P(t)}\frac{dP}{dt} = -x\frac{B}{\gamma}(1-\eta Bt)^{-1} \tag{P.16}$$

and

$$u_u = (L-x)\frac{1}{\gamma P(t)}\frac{dP}{dt} = (L-x)\frac{B}{\gamma}(1-\eta Bt)^{-1}. \tag{P.17}$$

Note that if η is positive then it appears that there is a potential for the pressure and velocities to blow up in finite time. We will see later however that the flame front reaches the end of the cylinder before that time.

From the equation governing the position of the flame front (S.13) we have

$$\frac{dS}{dt} = \left(u_u + \frac{m(t)}{\rho_u(t)}\right)\Bigg|_{S(t)} = (L-S(t))\frac{1}{\gamma P(t)}\frac{dP}{dt} + \frac{m(t)}{\rho_0}P^{-\frac{1}{\gamma}}(t). \tag{P.18}$$

This can be solved by using the solution for $P(t)$, giving

$$S(t) = L + \frac{L}{\gamma\rho_0 q_0}(1-\eta Bt)^{\frac{1-\gamma}{\gamma\eta}} + \left(S_0 - L - \frac{L}{\gamma\rho_0 q_0}\right)(1-\eta Bt)^{\frac{1}{\gamma\eta}}, \tag{P.19}$$

where the constant S_0 is the initial position of the flame front. One can easily determine that the time t_L when the flame front reaches the end of the cylinder is

$$t_L = \frac{1-\left((L-S_0)\frac{\gamma\rho_o q_0}{L}+1\right)^{-\eta}}{\eta B}. \tag{P.20}$$

Note that since $L > S_0$ it follows that

$$(L-S_0)\frac{\gamma\rho_0 q_0}{L} + 1 > 1$$

so that if $\eta > 0$ then $\frac{1}{B\eta} > t_L > 0$ and if $\eta < 0$ then, since both the numerator and denominator are negative t_L is also positive. Thus the flame front always reaches the end of the cylinder and P, u_b and u_u remain finite.

Now that the pressure and the velocity of the burnt gas are known the density of the burnt gas is found by solving equation (S.9) with the initial condition determined by the jump condition (S.11). The important thing to note is that ρ_b is a function of x and t. Otherwise we would have

$$0 = \frac{D}{Dt}(\rho_b^\gamma/P) = \frac{d}{dt}(\rho_b^\gamma/P) = \frac{d}{dt}(\rho_b^\gamma/P)\Big|_{S(t)} = \frac{d}{dt}\left(\left(\frac{\rho_0}{1+q_0\rho_0 P^{\frac{1}{\gamma}}}\right)^\gamma\right)$$

which is false because $\frac{dP}{dt} \neq 0$. The temperature field on both sides of the flame front is then found by using the ideal gas law.

Circular Flame Fronts. In the following examples we consider circular flame fronts in two dimensions. We will use $r-\theta$ coordinates with corresponding velocities v_r and v_θ, where positive v_θ denotes counterclockwise rotation. Denoting the radius of the circular flame front by $R(t)$ the mass flux is given by

$$m_i(t) = Q_1 P^{1+\eta}(t) - Q_2 \frac{P^{\frac{1}{\gamma}}(t)}{R(t)} \tag{C.1}$$

if the unburnt gas is in the region $r > R(t)$, and by

$$m_o(t) = Q_1 P^{1+\eta}(t) + Q_2 \frac{P^{\frac{1}{\gamma}}(t)}{R(t)} \tag{C.2}$$

if the unburnt gas is in the region $r < R(t)$. The subscripts i and o refer to the direction of the mass flux, inward or outward.

In all cases considered the flow will be independent of θ.

CASE I: Unbounded domain We first consider unbounded domains with, in general, a circular shear layer. As we will see the angular and radial velocities decouple so that the radial velocity is not effected by the angular velocity. As in the case of plane waves for an unbounded domain we have constant pressure and density as well as a divergenceless velocity field. Specifically, we have

$$\begin{aligned} P(t) &= 1 \\ \rho_u(t) &= \rho_0 = \frac{1}{1-q_0} \\ \rho_b(t) &= \frac{\rho_0}{1+q_0\rho_0} = 1 \end{aligned} \tag{C.3}$$

and

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} = 0.$$

The divergence equation gives

$$v_r = \frac{a(t)}{r} \tag{C.4}$$

for some arbitrary function $a(t)$.

There are two subcases to consider since we can either have the unburnt gas inside or outside the flame front. we will first discuss the subcase for which the unburnt gas is

outside the flame front, i.e. in the region $r > R(t)$ so we have a circular flame expanding outward. Imposing finite velocity at $r = 0$ gives

$$v_{r_b} = 0. \tag{C.5}$$

Noting that

$$S_n = \frac{dR}{dt}$$

we see that equation (S.6) gives, using $\rho_b = 1$,

$$\frac{dR}{dt} = \rho_0 \left(\frac{dR}{dt} - v_{r_u} \right) = m_i(t) = Q_1 - \frac{Q_2}{R(t)} \tag{C.6}$$

which can be solved to give an implicit formula for $R(t)$,

$$t = \frac{1}{Q_1}(R - R_0) + \frac{Q_2}{Q_1^2} ln \left(\frac{Q_1 R - Q_2}{Q_1 R_0 - Q_2} \right) \tag{C.7}$$

where R_0 is the position of the flame front at $t = 0$. We assume that R_0 is large enough so that $\frac{dR}{dt} > 0$. The expression (C.1) for m_i is after all an approximation for the mass flux and is not valid if R is to small.

It is now an easy matter to determine v_{r_u}. First, using (C.6) we see that $\frac{dR}{dt} = m_i(t)$ so that

$$v_{r_u}|_{R(t)} = m_i - \frac{m_i}{\rho_u} = q_0 m_i(t). \tag{C.8}$$

Evaluating (C.4) at $r = R(t)$ determines $a(t)$. The end result is

$$v_{r_u}(r,t) = \frac{R(t)}{r} v_{r_u}|_{R(t)} = \frac{q_0}{r}(Q_1 R(t) - Q_2). \tag{C.9}$$

We now turn to the angular velocity field. Let $\bar{v}_{\theta_u}(r)$ be the initial angular velocity in the unburnt gas. The momentum equation for the angular velocity is, since the flow is independent of θ,

$$\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_r v_\theta}{r} = 0. \tag{C.10}$$

We define $x_{r_o}(t)$ to be the radial position at time t of a particle of gas which had radial position r_0 at time $t = 0$. The function $x_{r_o}(t)$ satisfies

$$\begin{aligned} \frac{dx_{r_o}}{dt} &= v_r(x_{r_o}, t) \\ x_{r_o}(0) &= r_0. \end{aligned} \tag{C.11}$$

Then, by equation (C.10),

$$\frac{d}{dt}\left(x_{r_o}(t)v_\theta(x_{r_o}(t),t)\right) = v_r v_\theta + x_{r_o}\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r}\right)$$
$$= 0.$$

Thus, the differential equation (C.10) implies that the angular momentum is constant, i.e.

(C.12) $$x_{r_o}(t)v_\theta(x_{r_o}(t),t) \qquad \text{is constant}$$

Solving equations (C.11) in the region $r > R(t)$ we have

$$\frac{dx_{r_o}}{dt} = v_{r_u}(x_{r_o}(t),t) = \frac{a(t)}{x_{r_o}(t)} = \frac{R(t)}{x_{r_o}(t)}v_{r_u}(R(t),t)$$
$$= \frac{R(t)}{r(t)}\frac{dR}{dt}\left(1-\frac{1}{\rho_u}\right)$$
$$= q_0\frac{R(t)}{x_{r_o}(t)}\frac{dR}{dt}$$

so that, using the initial conditions $R(0) = R_0$ and $r(0) = r_0$ we have

(C.13) $$r^2(t) - r_0^2 = q_0(R^2(t) - R_0^2(t)).$$

Thus the angular velocity in the unburnt gas is given by

(C.14) $$v_{\theta_u}(r,t) = \frac{r_0}{r}\bar{v}_{\theta_u}(r_0)$$

where

(C.15) $$r_0(r,t) = \sqrt{r^2 - q_0(R^2(t) - R_0^2)}.$$

To determine the angular velocity inside the flame front we note that because the radial velocity is zero in this region the angular velocity is independent of time. Define $T(r)$ to be the time when the flame front was at r. Then by continuity of the tangential velocity across the flame we have

(C.16)
$$v_{\theta_b}(r) = \bar{v}_{\theta_b}(r) \qquad \text{for } r < R_0$$
$$v_{\theta_b}(r) = v_{\theta_u}(r, T(r)) \qquad \text{for } R_0 < r < R(t).$$

From equation (C.7) we have

(C.17) $$T(r) = \frac{1}{Q_1}(r - R_0) + \frac{Q_2}{Q_1^2}ln\left(\frac{Q_1 r - Q_2}{Q_1 R_0 - Q_2}\right).$$

We will now calculate the vorticity jump across the flame front. The vorticity is given by

$$\zeta = \frac{1}{r}\frac{\partial(rv_\theta)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta} = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}. \tag{C.18}$$

and we want to calculate

$$\delta\zeta(t) = \zeta_{u|(R(t),t)} - \zeta_{b|(R(t),t)}. \tag{C.19}$$

Now using equation (C.16) we have

$$\begin{aligned}\zeta_b &= \frac{v_{\theta_b}}{r} + \frac{\partial v_{\theta_b}}{\partial r} \\ &= \frac{v_{\theta_b}}{r} + \frac{\partial v_{\theta_u}}{\partial r}(r, T(r)) + \frac{\partial v_{\theta_u}}{\partial t}(r, T(r))\frac{\partial T}{\partial r}\end{aligned}$$

so that

$$\zeta_{b|(R(t),t)} = \frac{v_{\theta_b}(R(t),t)}{R(t)} + \frac{\partial v_{\theta_u}}{\partial r}(R(t),t) + \frac{\partial v_{\theta_u}}{\partial t}(R(t),t)\frac{\partial T}{\partial r}(R(t)).$$

By continuity of v_θ across the flame front we have

$$\delta\zeta = -\frac{\partial v_{\theta_u}}{\partial t}(R(t),t)\frac{\partial T}{\partial r}(R(t)). \tag{C.20}$$

Using equation (C.10) to replace the t derivative, noting that $\frac{\partial T}{\partial r} = \frac{1}{m_i(t)}$ and using the fact that $v_{r_u|R(t)} = q_0 m_i(t)$ leads to

$$\delta\zeta = q_0\left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r}\right)|_{R(t)} \tag{C.21}$$

This ends our discussion of the first subcase for which the unburnts gas was in the region $r > R(t)$.

We now take up the second subcase where the unburnt gas is confined to $r < R(t)$. As already noted, the pressure and densities are constant, given by equations (C.3). Equation (C.4) also holds so we have

$$v_{r_u} = 0. \tag{C.22}$$

The flame front position $R(t)$ and v_{r_b} are determined by

$$-\frac{dR}{dt} + v_{r_u} = -\rho_0\frac{dR}{dt} = m_o(t) = Q_1 + \frac{Q_2}{R(t)} \tag{C.23}$$

since now $\vec{n}$ is pointing inward, i.e. opposite the direction of $\vec{r}$. This leads to

$$v_{r_b} = \frac{q_0}{r}(Q_1 R(t) + Q_2) \tag{C.24}$$

where $R(t)$ is given implicitly by

$$t = -\frac{\rho_0}{Q_1}(R - R_0) + \frac{Q_{2\rho_o}}{Q_1^2} ln\left(\frac{Q_1 R + Q_2}{Q_1 R_0 + Q_2}\right). \tag{C.25}$$

We note that v_{r_b} is positive, as it must be because of the expansion of the gas upon being burnt.

The angular velocity of the unburnt gas is constant in time since $v_{r_u} = 0$. There are two regions in the burnt gas to consider; gas which was initially unburnt, region I, and gas which was initially burnt, region II. Let $r = B(t)$ be the boundary between these two regions.

First, we consider region II, Using conservation of angular momentum, equations (C.12), we have

$$v_{\theta_b}(r,t) = \frac{r_0(r,t)}{r}\bar{v}_{\theta_b}(r_0(r,t)) \tag{C.26}$$

where, again, $r_0(r,t)$ is the initial radial position of the gas which has radial position r at time t. The function r_0 is determined, as before, by solving

$$\begin{aligned} \frac{dx_{r_0}}{dt} &= v_{r_b}(x_{r_o}(t), t) \\ x_{r_o}(0) &= r_0 \end{aligned}$$

from which we obtain

$$r_0(r,t) = \sqrt{q_0\rho_0(R^2(t) - R_0^2) + r^2}. \tag{C.27}$$

Setting r_0 equal to R_0 and solving for r determines $B(t)$ as

$$B(t) = (1 + q_0\rho_0)R_0^2 - q_0\rho_0 R^2(t). \tag{C.28}$$

We now turn to region I, which contains gas which was initially unburnt. Let $T(r,t)$ be the time that the particle of gas which has radial position r at time t was at the flame front. Conservation of angular momentum and continuity of angular velocity gives

$$\begin{aligned} r v_{\theta_b}(r,t) &= R(T(r,t)) v_{\theta_u}\big(R(T(r,t)), T(r,t)\big) \\ &= R(T(r,t))\bar{v}_{\theta_u}\big(R(T(r,t))\big). \end{aligned}$$

We need to determine the function $R(T(r,t))$. This can be accomplished by solving

$$\frac{dr}{dt} = v_{r_b}(r(t),t)$$

along with the condition that at time $T(r,t)$ the particle was at the flame front, i.e.

$$r(T(r)) = R(T(r))$$

Thus we replace both r_0 and R_0 by $R(T(r,t))$ in equation (C.26) and solve for $R(T(r,t))$. This gives

$$R(T(r)) = \sqrt{\frac{r^2 + q_0\rho_0 R^2(t)}{1+q_0\rho_0}}. \tag{C.29}$$

Summarizing, we have

$$\begin{aligned} v_{\theta_u}(r,t) &= \bar{v}_{\theta_u}(r) && r < R(t) \\ v_{\theta_b}(r,t) &= \frac{r_0(r,t)}{r}\bar{v}_{\theta_b}(r_0(r,t)) && r > B(t) \\ v_{\theta_b}(r,t) &= \bar{v}_{\theta_u}\big(R(T(r,t))\big)\frac{R(T(r,t))}{r} && R(t) < r < B(t) \end{aligned} \tag{C.30}$$

where $r_0, B(t)$ and $R(T(r,t))$ given by (C.27), (C.28), and (C.29) respectively. Calculation of the vorticity jump proceeds as in the previous subcase and yields

$$\delta\zeta = \zeta_{u|R(t)} - \zeta_{b|R(t)} = q_0\left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r}\right)|_{R(t)} \tag{C.31}$$

which is the same formula for the previous subcase (see equation (C.21)).

CASE II: Bounded domain We next consider the effect of containing the circular flame front of the previous example in a closed region. Let the region Ω be a closed cylinder with length L and radius C. The flame front is concentric with the cylinder.

The pressure equation (S.2) gives

$$\frac{dP}{dt} = \frac{2q_0\gamma}{C^2}m(t)R(t) \tag{C.32}$$

and the divergence equation gives

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} = -\frac{1}{\gamma P(t)}\frac{dP}{dt}. \tag{C.33}$$

The latter equation has the general solution

$$v_r = -\frac{r}{2\gamma P(t)}\frac{dP}{dt} + \frac{a(t)}{r}. \tag{C.34}$$

Imposing the boundary conditions at $r = 0$ and at $r = C$ gives the radial velocity as

$$v_r = \begin{cases} -\frac{r}{2\gamma P(t)}\frac{dP}{dt} = -r\frac{q_0}{C^2}\frac{m(t)R(t)}{P(t)} & 0 < r < R(t) \\ \left(\frac{C^2}{r} - r\right)\frac{1}{2\gamma P(t)}\frac{dP}{dt} = \left(\frac{1}{r} - \frac{r}{C^2}\right)\frac{q_0 m(t)R(t)}{P(t)} & R(t) < r < C \end{cases} \tag{C.35}$$

For the case where the unburnt gas is in the region $r > R(t)$ so there is an expanding spherical flame then equation (S.6) gives

$$\frac{dR}{dt} = \frac{m_i(t)}{\rho_{b|R(t)}} - \frac{R(t)}{2\gamma P(t)}\frac{dP}{dt} = \frac{m_i(t)}{\rho_u} + \left(\frac{C^2 - R^2(t)}{R(t)}\right)\frac{1}{2\gamma P(t)}\frac{dP}{dt}. \tag{C.36}$$

Using equation (C.32) to remove the P derivative we obtain a coupled set of equations for P and R;

$$\begin{aligned} \frac{dP}{dt} &= \frac{2q_0\gamma}{c^2} m_i(t)R(t) \\ \frac{dR}{dt} &= \frac{m_i(t)}{\rho_u} + \left(1 - \frac{R^2(t)}{C^2}\right)\frac{q_0 m_i(t)}{P(t)}. \end{aligned} \tag{C.37}$$

A similar procedure for the case where the unburnt gas is in the region $r < R(t)$ gives

$$\frac{dR}{dt} = \frac{m_o(t)}{\rho_u} - \frac{R(t)}{2\gamma P(t)}\frac{dP}{dt} = -\frac{m_o(t)}{\rho_{b|R(t)}} + \left(\frac{C^2 - R^2(t)}{R(t)}\right)\frac{1}{2\gamma P(t)}\frac{dP}{dt} \tag{C.38}$$

and the coupled set of equations for P and R;

$$\begin{aligned} \frac{dP}{dt} &= \frac{2q_0\gamma}{C^2} m_o(t)R(t) \\ \frac{dR}{dt} &= -\frac{m_o(t)}{\rho_u} - \frac{q_0}{C^2}\frac{R^2(t)m(t)}{P(t)}. \end{aligned} \tag{C.39}$$

The mass fluxes m_i and m_o are given in terms of P and ρ_u by equations (C.1) and (C.2) respectively. In both cases ρ_u is given in terms of $P(t)$ by

$$\rho_u = \rho_o P^{\frac{1}{\gamma}}(t).$$

Note also the pressure increases monotonically and that $R(t)$ goes in the direction of the unburnt gas in both cases. Angular momentum is conserved (see equation (C.12), which still holds), so the angular velocity is calculated via

$$x_{r_o}(t)v_\theta(x_{r_o}(t), t) = x_{r_o}(t)v_\theta(x_{r_o}(\bar{t}), \bar{t}) \tag{C.40}$$

where x_{r_o} is the radial position a particle of gas at time t which had position r_0 at $t = 0$. Thus x_{r_0} is determined by

$$\begin{aligned} \frac{dx_{r_o}}{dt} &= v_r(x_{r_o}, t) \\ x_{r_o}(0) &= r_0. \end{aligned} \tag{C.41}$$

Solving this we obtain

$$\left(C^2 - x_{r_o}^2(t)\right) P^{\frac{1}{\gamma}}(t) = \text{ constant} \qquad \text{for } r > R(t) \tag{C.42}$$

$$x_{r_o}(t) P^{\frac{1}{2\gamma}}(t) = \text{ constant} \qquad \text{for } r < R(t) \tag{C.43}$$

For the unburnt gas we set $\bar{t} = 0$ and invert the equations (C.42) and (C.43) to give r_0 as a function of r and t to obtain

$$v_{\theta_u}(r, t) = \frac{r_0}{r} \bar{v}_{\theta_u}(r)$$

where

$$\begin{aligned} r_0(r,t) &= \sqrt{C^2 - (C^2 - r^2) P^{\frac{1}{\gamma}}(t)} && \text{unburnt gas in } r > R(t) \\ r_0(r,t) &= r P^{\frac{1}{2\gamma}}(t) && \text{unburnt gas in } r < R(t) \end{aligned} \tag{C.44}$$

In order to determine the angular velocity in the burnt gas we define $T(r, t)$ to be the time that a particle which has radial position r at time t was at the flame front. In terms of T the angular velocity of the burnt gas is given by, after using continuity of v_θ at the flame front,

$$v_{r_b}(r, t) = \frac{R(T(r,t))}{r} v_{\theta_u}\big(R(T(r,t)), T(r,t)\big). \tag{C.45}$$

The function T is given implicitly by

$$\begin{aligned} r P^{\frac{1}{2\gamma}}(t) &= R(T(r,t)) P^{\frac{1}{2\gamma}}(T(r,t)) && \text{unburnt gas in } r > R(t) \\ (C^2 - r^2) P^{\frac{1}{\gamma}}(t) &= \left(C^2 - R^2(T(r,t))\right) P^{\frac{1}{\gamma}}(T(r,t)) && \text{unburnt gas in } r < R(t). \end{aligned} \tag{C.46}$$

Note that in both cases the right had side is strictly monotonic and hence we can invert and find T. Also $T(R(t), t) = t$. Differentiating w.r.t. r and evaluating at $r = R(t)$ we obtain

$$1 = \left(\frac{dR}{dt} \frac{\partial T}{\partial r} - v_{r_b} \frac{\partial T}{\partial r} \right) \Big|_{R(t)} \tag{C.47}$$

and so

$$\frac{\partial T}{\partial r}\Big|_{R(t)} = \frac{1}{\frac{dR}{dt} - v_{r_b}}. \tag{C.48}$$

This is true in both cases.

Calculation of $\delta\zeta = \zeta_{u|R(t)} - \zeta_{b|R(t)}$ then goes as follows. From (C.45) we have

$$\begin{aligned}
\zeta_b &= \frac{v_{\theta_b}}{r} + \frac{\partial v_{\theta_b}}{\partial r} \\
&= \frac{v_{\theta_b}}{r} + \left[-\frac{v_{\theta_b}}{r} + \frac{1}{r}\left(\frac{dR}{dt}\frac{\partial T}{\partial r} v_{\theta_u} + R\frac{\partial v_{\theta_u}}{\partial r}\frac{dR}{dt}\frac{\partial T}{\partial r} + R\frac{\partial v_{\theta_u}}{\partial t}\frac{\partial T}{\partial r} \right) \right] \Big|_{t=T(r,t)} \\
&= \frac{1}{r}\left[R\frac{dR}{dt}\frac{\partial T}{\partial r}\left(\frac{v_{\theta_u}}{R} + \frac{\partial v_{\theta_u}}{\partial r} \right) + R\frac{\partial T}{\partial r}\frac{\partial v_{\theta_u}}{\partial t} \right] \Big|_{t=T(r,t)}.
\end{aligned}$$

Using the θ-component of the momentum equation to eliminate the time derivative of v_{θ_u} we have

$$\zeta_b = \frac{1}{r}\left[R\frac{dR}{dt}\frac{\partial T}{\partial r}\left(\frac{v_{\theta_u}}{R} + \frac{\partial v_{\theta_u}}{\partial r} \right) - Rv_{r_u}\frac{\partial T}{\partial r}\left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r} \right) \right] \Big|_{t=T(r,t)}$$

Evaluating this at $r = R(t)$ we obtain

$$\begin{aligned}
\zeta_{b|R(t)} &= \left(\frac{dR}{dt} - v_{r_u} \right) \frac{\partial T}{\partial r}\left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r} \right) \Big|_{R(t)} \\
&= \left(\frac{\frac{dR}{dt} - v_{r_u}}{\frac{dR}{dt} - v_{r_b}} \right) \left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r} \right) \Big|_{R(t)} \\
&= \frac{\rho_b}{\rho_u}\left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r} \right) \Big|_{R(t)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\delta\zeta &= \left(1 - \frac{\rho_b}{\rho_u} \right) \left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r} \right) \Big|_{R(t)} \\
&= \frac{q_0\rho_b}{P}\left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r} \right) \Big|_{R(t)}.
\end{aligned} \tag{C.49}$$

Note that the term in brackets is just the vorticity of the unburnt gas at the flame front. Using equation (S.11) which relates ρ_b and ρ_u at the flame front we see that

$$\zeta_{b|R(t)} = \frac{P}{P + q_0\rho_u}\zeta_{u|R(t)}. \tag{C.50}$$

Thus, the vorticity is reduced behind the flame, due to the decrease in density.

Discussion. Now that we have solved the equations for a variety of simple situations we can discuss some of the effects of curvature and containment of the gas.

First we consider the effects of containment. Containment of the gas in a bounded region results in a buildup of the pressure $P(t)$ and of the densities. It also results in the burnt gas becoming nonuniform. This increase in the pressure results in an increase in the mass flux, since $\eta + 1 > 0$. From equation (S.6) we find that the flame speed w.r.t. the unburnt gas decreases as the pressure increases, provided that γ is less than 2 which is always the case.

The effects of incorporating a curvature dependence of the mass flux is to reduce/increase the mass flux if the flame front is convex/concave w.r.t. the unburnt gas. This results in a corresponding decrease/increase of the flame velocity w.r.t. unburnt gas.

The effects on the vorticity jump of enclosing the gas in a finite volume as well as the effects of curvature dependence of the mass flux are all contained in equation (C.49), namely

$$\delta\zeta = \frac{q_0\rho_b}{P}\left(\frac{\partial v_{\theta_u}}{\partial r} + \frac{v_{\theta_u}}{r}\right)\Big|_{R(t)} \tag{C.51}$$

In all cases, the expansion of the gas behind the flame front reduced the vorticity. Note that as the size of the containment vessel becomes infinite both ρ_b and P approach 1 and the vorticity jump for a circular flame front in an unbounded domain is recovered. Next by letting the value or r go to infinity the vorticity jump for a planar flame front in an unbounded domain is recovered as well. We also note that in a confined chamber

$$\frac{\rho_b}{P}\Big|_{R(t)} = \frac{1}{q_0 + \rho_0 P^{1-\frac{1}{\gamma}}}$$

decreases with time since $\gamma > 1$ so the vorticity jump is less prominent in confined circular geometries.

We end our discussion with the following remark: For flame fronts moving in two space dimensions, there is a general procedure to compute the vorticity field. In both burnt and unburnt gas, we have the equation, $\frac{D}{Dt}\left(\frac{\omega}{\rho}\right) = 0$ while the vorticity jump across the flame front is determined from the general formula of Hayes ([20]). The formulas for the vorticity jump given explicitly in the simple geometries are the same as those computed through this general formula.

§5: Some Mathematical Problems in Zero Mach Number Combustion with Strong Heat Release.

Here we give a brief discussion of important open problems regarding the equations for zero Mach number combustion.

Problem # 1 Find interesting conditions on the initial data so that Embid's local existence theorems ([7], [8], [9]) for (0.2) and (1.48) extend to global existence theorems — perhaps, as suitable weak solutions.

Problem # 2 Follow ideas of Klainerman and Majda ([21]) and extend Schochet's work ([10]) to give a rigorous derivation of the next term in the low Mach number asymptotics. This term involves linear equations for acoustical noise at low Mach numbers — an important practical topic ([22]). Also, extend Schochet's derivation from the inviscid case to allow for fixed non-zero Prandtl and Lewis numbers for the equations in (1.1).

Problem # 3 Formulate and prove a suitable global existence theorem for the $1-D$ integro-differential reaction diffusion system in (3.8) and (3.3). Also consider other boundary conditions that allow for fuel injection. Find conditions on the piston motion, $\overline{X}(t)$, which guarantee this existence theorem. For appropriate boundary conditions for T, Z and suitable time periodic piston motions, $\overline{X}(t)$, decide whether the system of equations in (3.8) and (3.3) has time periodic solutions. Are chaotic dynamic motions of the equations in (3.8) and (3.3) possible with periodic piston motions $\overline{X}(t)$?

Problem # 4 Using high activation energy asymptotics, provide rigorous justification (even for short times !) for the one-dimensional flame sheet equations in (3.10) including a formula for the mass flux, $m(\hat{q}, t')$. This is a difficult but important first step in providing a rigorous understanding of the formal limit in section 2.

Problem # 5 Find other interesting exact solutions of the flame front free boundary equations in (2.17) besides those presented in Section 4. A systematic linear and nonlinear stability analysis of the exact solutions in section 4 which delineates the role of curvature, confinement, and vorticity production on the stability of solutions would be very interesting.

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