SIMPLE RESONANCE REGIONS OF TORUS DIFFEOMORPHISMS

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Abstract. This paper discusses resonance regions for two parameter families of diffeomorphisms and vector fields on the two dimensional torus. Resonance regions with at most two resonant periodic orbits (in the discrete case) or two equilibrium points (for flows) are studied. We establish global geometric properties of these regions with topological arguments.

Key words. dynamical systems, bifurcation theory

AMS(MOS) subject classifications.

This paper discusses the geometry of the simplest resonance regions displayed by three weakly coupled oscillators. Much more extensive descriptions of the dynamics of three coupled oscillators are given in Baesens, Guckenheimer, Kim and Mackay [1]. Indeed, the results presented here are a footnote to [1] and the reader should look to this work for diagrams that illustrate the phenomena discussed here. Generic two parameter families of three weakly coupled oscillators give vector fields in which there is a smoothly varying family of invariant three dimensional tori. By taking cross-sections of the flows on the invariant three dimensional tori, the three interacting frequencies can be studied via a family of diffeomorphisms of the two torus. This is the setting in which we shall work.

The theory of diffeomorphisms of the circle is thoroughly developed. It yields information about resonances and mode lockings of two interacting frequencies. There are stringent limits on the dynamical possibilities. The dynamical behavior of a circle diffeomorphism can be summarized via its rotation number ρ. If ρ is irrational, then the motion is quasiperiodic. Otherwise the rotation number is rational, and all trajectories tend to periodic orbits of the same period in both forwards and backwards time.

The theory of diffeomorphisms of the two torus is much more complicated than the theory of diffeomorphisms of the circle. Phase portraits on a two dimensional torus can have multiple attractors as well as chaotic trajectories. Forward and backward limit sets of different types of different types can coexist. The three types of limit sets that are seen most frequently in generic diffeomorphisms are periodic orbits, invariant curves and the whole two torus. Examples with chaotic trajectories occur, but they are seldom found without a systematic search. A natural setting for the study of diffeomorphisms of the two torus is the investigation of families in which the amount of rotation in the two directions

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around the torus can be varied. The family of diffeomorphisms $f_\Omega(x) = x + \Omega$ is called the family of **translations**. Here the two dimensional parameter Ω varies the amount of rotation. Most of the theory that has been developed thus far has been within the context of families of the form $f_\Omega(x) = x + \Omega + \epsilon g(x)$ with $g(x)$ a nonlinear, doubly periodic function and ϵ a third parameter which controls the size of nonlinearity. For fixed ϵ, the system goes through regions with many different rotation vectors as Ω is varied.

One can begin to understand the dynamics of a generic family of diffeomorphisms of the two torus by first considering the family of rotations and then perturbations of this family. In a generic two-parameter family that is close to the family of rotations, KAM theory guarantees that the sets of parameters values for which the corresponding diffeomorphisms are not smoothly equivalent to irrational rotations occupies a small measure in the parameter space. The diffeomorphisms with limit sets that are periodic orbits or invariant curves are the object of our interest. We recall some properties of the parameter sets giving rise to this mode-locked behavior and of the dynamics that are observed on the torus when the mode-locking occurs.

In the product of the parameter and the phase spaces, there is a two torus of periodic orbits corresponding to each rational parameter vector $(p_1/q_1, p_2/q_2)$. The period q of these orbits is the least common multiple of q_1 and q_2. It is easy to check that this torus is a nonsingular set of roots of the pair of equations $f^q_\Omega(x) = x$ on the product space. It follows that a family of torus diffeomorphisms that are near rotations will have a torus P of periodic orbits of period q. Unlike the family of translations, one cannot expect that P will project to a single point in the parameter space. The image of the projection is called the **resonance region** of the rotation vector with $(p_1/q_1, p_2/q_2)$ associated with P. Basic properties of resonance regions are described in [3]. As the image of the map of a torus into the plane, there are topological constraints on the form of the resonance region and its boundary [4]. In particular, generic families have resonance regions whose only singularities are cusp points. Moreover, a resonance region cannot have a connected boundary without cusp points. The presence of cusp points implies there are parameter regions in which there are four periodic orbits with rotation vector $(p_1/q_1, p_2/q_2)$.

There are resonance regions of families of diffeomorphisms for which the individual systems contain only one or two periodic orbits of the resonance rotation vector. These resonance regions appear to arise naturally in two ways. First, if one starts with a "Mathieu" family in which the nonlinear part $\epsilon g(x)$ of the family $f_\Omega(x) = x + \Omega + \epsilon g(x)$ is given by a pair of trigonometric monomials, then the expansion of $f^n(x) - x$ near a resonance region of order n as a power series in ϵ will normally have dominant and subdominant terms that are trigonometric monomials. Galkin [2] describes the combinatorics associated with determining which are the dominant terms and their degrees. The new results of this paper address the geometry of these "simplest" resonance regions. The global topology of the torus plays a prominent role in the discussion.

The behavior associated with invariant curves is more complicated. For each rotation vector $\Omega = (\omega_1, \omega_2)$ satisfying only one independent equation $p_1\omega_1 + p_2\omega_2 = q$ with integer coefficients, there is also a two dimensional torus in the product of the phase space and the parameter space for the family of translations consisting of orbits that have this rotation vector. The trajectories on this torus are quasiperiodic, but lie on closed curves whose homotopy type is determined by Ω. For perturbations, these tori may or not persist: KAM theory provides the techniques proving persistence for "good" rotation vectors. As Ω varies over vectors satisfying the relation $p_1\omega_1 + p_2\omega_2 = q$, one obtains a line in the parameter space for the family of translations along which there are invariant curves of the same homotopy type for the corresponding diffeomorphisms. For perturbations of the family of translations, the first approximation to the set of parameters with invariant curves of the specified homotopy type is a strip. As one moves along this strip, there are many points at which the corresponding diffeomorphism acts like an irrational rotation on a smooth invariant curve. At other points of the parameter space, there may be invariant curves with less smoothness that are formed from saddle separatrices. There may also be holes in the strips in which there is no closed invariant curve at all. In addition to the **rotational** invariant curves that have non-trivial homotopy type, one encounters invariant curves that can be continuously deformed to a point on the torus.

The basic framework of a set of overlapping resonance regions and strips associated with invariant curves of specific homotopy types can be substantially refined by consideration of bifurcations showing the behavior at the boundaries of the strips and resonance regions and the transitions that take place inside them. The geometry associated with these bifurcations is complex, but there are several basic facts and principles that can be used to understand this geometry.

First, one has the simple fact that two closed curves of different non-trivial homotopy types on the two dimensional torus must intersect. This implies that if rotational invariant curves of different homotopy types exist, they are singular curves that contain periodic orbits. Consequently, strips associated with different homotopy types intersect only in resonance regions.

Second, there is an empirical observation that phase portraits of torus diffeomorphisms near translations usually look similar to those that come from the time one map of flows. This observation is supported by the fact that every torus diffeomorphism has finite Taylor expansions that occur as the Taylor expansions of time 1 maps of flows. Since flows on two dimensional manifolds do not have chaotic trajectories, examining the dynamics of families of time one maps of flows gives us a gentler approach to the study of resonance phenomena for diffeomorphisms. Nonetheless, many of the topological features associated with diffeomorphisms cannot occur in time one maps of flows, so additional analysis is required once one has a good intuition for the dynamics of families of flows.

Third, there is a list of codimension one and two bifurcations that occur in generic two parameter families of two dimensional flows. This list can be used to interpret and guide numerical explorations that seek to delineate the dynamics that occur in specific examples.

The interplay between the theoretical analysis of normal forms and interactive computations of phase portraits has played an important role in our developing understanding of families of torus maps. The novel aspect in the analysis of normal forms for our studies has been the necessity of looking carefully at more complicated patterns of saddle connections than have been examined previously.

Recall some of the different types of codimension one and two bifurcations for flows on two dimensional manifolds. In codimension one, there are saddle-node bifurcations at which a pair of equilibrium points coalesce and Hopf bifurcations at which a family of limit cycles ends at an equilibrium. Pairs of limit cycles can coallesce in bifurcations called either double limit cycles or saddle-nodes of periodic orbits. The final type of codimension one bifurcation involves a homoclinic or heteroclinic orbit in which there is a trajectory that terminates at a saddle point as time tends to $\pm\infty$. In codimension two, the list of bifurcations is longer. The local bifurcations involving equilibrium points are cusps, degenerate Hopf bifurcations and the Takens-Bogdanov bifurcations at which an equilibrium has a nilpotent linear part. In addition, there are cusps of limit cycles and various types of bifurcations involving homoclinic and heteroclinic trajectories. For understanding the geometry of the simplest resonance regions occurring in two parameter families, the additional bifurcations that occur involve homoclinic orbits to saddle points with eigenvalues of equal magnitude, saddle-node loops in which there is a homoclinic trajectory for a saddle-node, and double saddle-loops in which a saddle point has two distinct homoclinic trajectories.

Consider a two parameter family of flows on the two dimensional torus with the property that each flow contains at most two equilibrium points. We want to prove as many properties about the bifurcation diagram of such a family as possible.

PROPOSITION. *Let $f : T^2 \to R^2$ be a smooth map with generic singularities and the property that, for each $y \in R^2$, $f^{-1}(y)$ is zero, one or two points. Then the image of f is an annulus with smooth boundaries. The singular set of the map consists of two curves that lie in the same non-trivial homotopy class.*

Proof. The generic singularities of maps between two dimensional manifolds are folds, cusps and the transversal intersection of two fold curves. Since cusps have an adjacent region which is triply covered by the map, cusps do not occur. Nor can there be intersections of two fold curves since each fold has an adjacent region for which there are at least two preimages of each image point. Thus, adjacent to a transversal intersection of two folds is a region in which each point has at least four preimages. We conclude that the singular image of the map is a smooth one dimensional manifold consisting entirely of folds. Thus, the singular image divides the plane into a finite number of regions, one of these regions is the regular image of the map, and the regular image is double covered. Since the Poincaré index of the torus is zero, it follows that there are precisely two boundary components to the image. These components divide the torus into homeomorphic regions. Hence the singular set on the torus must consist of two curves that lie in the same non-trivial homotopy class.

THEOREM. *Let $\dot{x} = \Omega + g(x)$ define a two parameter family of flows on the torus with the property that for each Ω there are at most two equilibrium points of the corresponding flow.*

(1) *The resonance region for the map is an annulus.*

(2) *The flows in the bounded component of the complement of the resonance region (the "hole") do not have a cross-section. In the notation of [1], the flow has type D_ω for a fixed rational ω.*

(3) *There are at least two curves of Hopf bifurcations and six curves of homoclinic bifurcations in the resonance region.*

(4) *The curves of homoclinic bifurcations meet in at least two necklace points at which all of the saddle separatrices of the flow are homoclinic.*

Proof. The first statement is a consequence of the previous proposition since the resonance region is the image of the map $-g : T^2 \rightarrow R^2$. The existence of at least two Takens-Bogdanov points on each boundary of the resonance region was proved in an earlier paper [3]. This implies the existence of at least two curves of Hopf bifurcations and two curves of saddle-node bifurcations. The points on the torus at which the map $-g : T^2 \rightarrow R^2$ has a fold give rise to the saddle-node bifurcations for appropriate parameter values (that depend on the point). Let C be the curve on the torus whose image forms the inner component of the boundary of the image of $-g$. Then C is a smooth closed curve that is homotopically non-trivial on the torus. Observe that if Ω is a parameter value in the hole, then the image of C by the map $-g$ has winding number ± 1 with respect to Ω. Thus, the map $g + \Omega : T^2 \rightarrow R^2$ has image that is an annulus, the origin is in the bounded region of its complement and the image of C has winding number ± 1 with respect to the origin. We assert that this is incompatible with the presence of a cross-section to the flow.

If a flow has a cross-section, then the winding number of the vector field with respect to the origin is zero along any closed curve on the torus. This is readily seen by constructing a homotopy of the vector field to a constant vector field through vector fields without fixed points. Winding vectors do not change under such homotopies. The construction of the homotopy can be accomplished by an initial homotopy that makes the vector field have a constant return time. It is then transverse to a one parameter family of cross-sections. A second homotopy of coordinate changes makes these cross-sections parallel to one another. Finally, a homotopy of the vector field makes it of unit length and orthogonal to the parallel cross-sections. It is then constant. We conclude that $g(x) + \Omega$ has no cross-section.

Consider the vector field associated to a parameter value inside the hole of the resonance region. There are at least two periodic orbits with opposite orientations, but we can say more as well. The torus is divided into annuli by the periodic orbits. Each of these annuli has either trajectories whose orientation does not change or trajectories whose orientation changes by $\pm\pi$. The second type of annulus is called a **Reeb component**. The sum of the changes in orientation from all the annuli is $\pm 2\pi$, corresponding to the index change of the vector field as we traverse the curve along which folds occur in the vector field. If we collapse the annuli in which orientation does not change onto closed curves, we obtain a topological flow on a torus with periodic orbits separated by Reeb components. At least one of the periodic orbits must be attracting (and hence one repelling) since otherwise the sum of changes in orientation from the Reeb components would be 0.

The flow of vector fields for parameters in the hole have annuli that are attracting their boundaries contained in the Reeb components. The presence of an attracting annulus with boundary in Reeb components is a persistent phenomenon under perturbation. Though the size of an attracting annulus may change discontinuously as periodic orbits undergo bifurcation, perturbations of the flow will have an attracting annulus that intersects the original one. As our parameter varies over the disk, we can therefore find continuous families of closed curves that lie in the the Reeb components that contain the boundaries of an attracting annulus.

We next consider the geometry of the vector fields obtained by traversing the parameter curve γ bounding the hole. For each parameter value, there is still a periodic orbit of the homotopy type of those found in the hole since a saddle-node bifurcation will not affect periodic orbits of opposite orientations. Observe that the curve σ traversed on the torus by the saddle-node points lies in a different (non-trivial) homotopy type from the periodic orbits since the winding number of the vector field along this curve is non-zero for parameter values inside the hole. We also assert that the saddle-node points cross Reeb components containing the boundary of an attracting annulus as one traverses γ. If this were not the case, then a continuous family of closed curves in the Reeb components would undergo a non-trivial homotopy along σ. This is a contradiction since these families of closed curves in the Reeb components can be extended continuously over the hole, implying that the homotopy is trivial.

Since the saddle-nodes cross Reeb components as parameters traverse γ, there must be parameter values in γ for which the saddle-node points cross the boundaries of Reeb components. In a generic family of vector fields, these must be codimension two bifurcations which adjoin regions with different numbers of periodic orbits. Examining the list of codimension two bifurcations for families of vector fields with at most one saddle point, the only possibility for these bifurcations are that they are **saddle-node loops**. If the saddle-node has a negative eigenvalue, this means that the unstable separatrix of the saddle-node lies on the boundary of the stable manifold of the saddle-node. Pursuing this logic further, one finds that there must be at least four such bifurcation points corresponding to the boundary components of two distinct Reeb components. ¿From each of these codimension two bifurcations emerges a curve of homoclinic bifurcations in the parameter space. These bifurcation curves are distinct from those that end at the Takens-Bogdanov points since the homoclinic orbits are homotopically non-trivial. We conclude that there are at least six curves of homoclinic bifurcations in our family of vector fields. This ends the proof of the theorem.

REMARKS. The homoclinic bifurcation curves can end only at saddle-node loops in generic families with at most one saddle point. The arguments given above can be extended to show that the homoclinic bifurcation curves that we have identified must cross the resonance region from one boundary component to the other. If one tracks periodic orbits along a homotopically non-trivial closed curve in the resonance region, they follow paths that are homotopic to σ. Thus the saddle-ponts cross "Reeb components" as one traverses the parameter curve. The changes of behavior that occur in the position of saddle points correspond to homoclinic bifurcations. Since the stability of the two boundary components of a Reeb component differ, there must be codimension two bifurcations at which the rotational homoclinic orbits come from a trace zero saddle point. Generically, there are curves of saddle-node bifurcations for periodic orbits that emanate from such codimension two bifurcations. Further analysis of the order with which different homoclinic bifurcations will be encountered on the two boundary components of a Reeb component indicates that the homoclinic bifurcations cross. Thus there are codimension two bifurcation points with double homoclinic cycles. These bifurcations are called **necklace** points in [1].

When one considers the simplest resonance regions for diffeomorphisms rather than flows, the structures described above become more complicated. Curves of homoclinic bifurcations become thickened and the structure of saddle-nodes of periodic orbits becomes more complicated, with the addition of "Chenciner bubbles" where a pair of resonant invariant curves try to approach each other. The reader is referred to [1] for a discussion of what we know about the details of these additional complications that appear in the bifurcation diagrams of two parameter families of diffeomorphisms of the two dimensional torus.

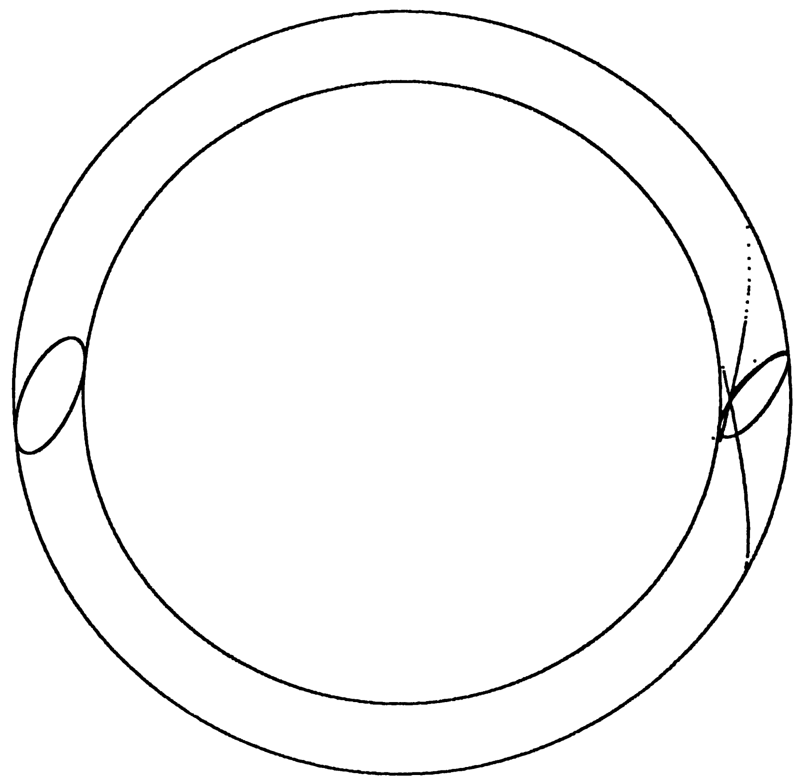

FIGURE 1: A numerically computed simple resonance region for the family of flows

$$\dot{x} = \Omega_x + \cos(2\pi x) + 0.1\cos(2\pi y)$$
$$\dot{y} = \Omega_y + \cos(2\pi x) + 0.1\cos(2\pi y)$$

The entire resonance region with its boundary of saddle-node curves is shown. Half of each small oval is a curve of Hopf bifurcations and half is a curve along which there is a saddle point with determinant one. Curves of parameters with homoclinic bifurcations are shown on one side of the resonance region.

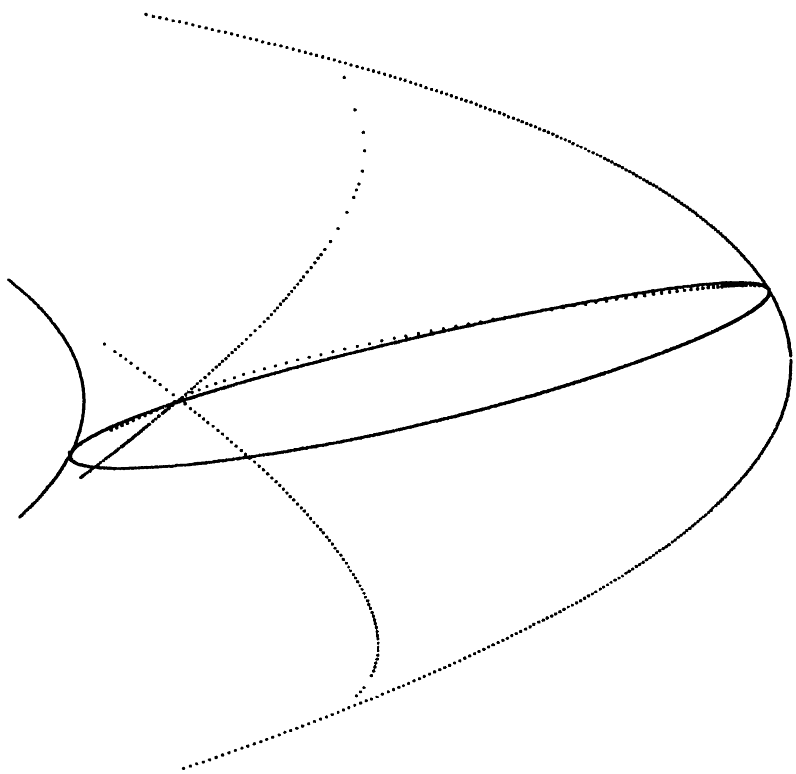

FIGURE 2: An enlarged view of one side of the simple resonance region shown in Figure 1.

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around the torus can be varied. The family of diffeomorphisms $f_\Omega(x) = x + \Omega$ is called the family of **translations**. Here the two dimensional parameter Ω varies the amount of rotation. Most of the theory that has been developed thus far has been within the context of families of the form $f_\Omega(x) = x + \Omega + \epsilon g(x)$ with $g(x)$ a nonlinear, doubly periodic function and ϵ a third parameter which controls the size of nonlinearity. For fixed ϵ, the system goes through regions with many different rotation vectors as Ω is varied.

One can begin to understand the dynamics of a generic family of diffeomorphisms of the two torus by first considering the family of rotations and then perturbations of this family. In a generic two-parameter family that is close to the family of rotations, KAM theory guarantees that the sets of parameters values for which the corresponding diffeomorphisms are not smoothly equivalent to irrational rotations occupies a small measure in the parameter space. The diffeomorphisms with limit sets that are periodic orbits or invariant curves are the object of our interest. We recall some properties of the parameter sets giving rise to this mode-locked behavior and of the dynamics that are observed on the torus when the mode-locking occurs.

In the product of the parameter and the phase spaces, there is a two torus of periodic orbits corresponding to each rational parameter vector $(p_1/q_1, p_2/q_2)$. The period q of these orbits is the least common multiple of q_1 and q_2. It is easy to check that this torus is a nonsingular set of roots of the pair of equations $f^q_\Omega(x) = x$ on the product space. It follows that a family of torus diffeomorphisms that are near rotations will have a torus P of periodic orbits of period q. Unlike the family of translations, one cannot expect that P will project to a single point in the parameter space. The image of the projection is called the **resonance region** of the rotation vector with $(p_1/q_1, p_2/q_2)$ associated with P. Basic properties of resonance regions are described in [3]. As the image of the map of a torus into the plane, there are topological constraints on the form of the resonance region and its boundary [4]. In particular, generic families have resonance regions whose only singularities are cusp points. Moreover, a resonance region cannot have a connected boundary without cusp points. The presence of cusp points implies there are parameter regions in which there are four periodic orbits with rotation vector $(p_1/q_1, p_2/q_2)$.

There are resonance regions of families of diffeomorphisms for which the individual systems contain only one or two periodic orbits of the resonance rotation vector. These resonance regions appear to arise naturally in two ways. First, if one starts with a "Mathieu" family in which the nonlinear part $\epsilon g(x)$ of the family $f_\Omega(x) = x + \Omega + \epsilon g(x)$ is given by a pair of trigonometric monomials, then the expansion of $f^n(x) - x$ near a resonance region of order n as a power series in ϵ will normally have dominant and subdominant terms that are trigonometric monomials. Galkin [2] describes the combinatorics associated with determining which are the dominant terms and their degrees. The new results of this paper address the geometry of these "simplest" resonance regions. The global topology of the torus plays a prominent role in the discussion.

The behavior associated with invariant curves is more complicated. For each rotation vector $\Omega = (\omega_1, \omega_2)$ satisfying only one independent equation $p_1\omega_1 + p_2\omega_2 = q$ with integer coefficients, there is also a two dimensional torus in the product of the phase space and the parameter space for the family of translations consisting of orbits that have this rotation vector. The trajectories on this torus are quasiperiodic, but lie on closed curves whose homotopy type is determined by Ω. For perturbations, these tori may or not persist: KAM theory provides the techniques proving persistence for "good" rotation vectors. As Ω varies over vectors satisfying the relation $p_1\omega_1 + p_2\omega_2 = q$, one obtains a line in the parameter space for the family of translations along which there are invariant curves of the same homotopy type for the corresponding diffeomorphisms. For perturbations of the family of translations, the first approximation to the set of parameters with invariant curves of the specified homotopy type is a strip. As one moves along this strip, there are many points at which the corresponding diffeomorphism acts like an irrational rotation on a smooth invariant curve. At other points of the parameter space, there may be invariant curves with less smoothness that are formed from saddle separatrices. There may also be holes in the strips in which there is no closed invariant curve at all. In addition to the **rotational** invariant curves that have non-trivial homotopy type, one encounters invariant curves that can be continuously deformed to a point on the torus.

The basic framework of a set of overlapping resonance regions and strips associated with invariant curves of specific homotopy types can be substantially refined by consideration of bifurcations showing the behavior at the boundaries of the strips and resonance regions and the transitions that take place inside them. The geometry associated with these bifurcations is complex, but there are several basic facts and principles that can be used to understand this geometry.

First, one has the simple fact that two closed curves of different non-trivial homotopy types on the two dimensional torus must intersect. This implies that if rotational invariant curves of different homotopy types exist, they are singular curves that contain periodic orbits. Consequently, strips associated with different homotopy types intersect only in resonance regions.

Second, there is an empirical observation that phase portraits of torus diffeomorphisms near translations usually look similar to those that come from the time one map of flows. This observation is supported by the fact that every torus diffeomorphism has finite Taylor expansions that occur as the Taylor expansions of time 1 maps of flows. Since flows on two dimensional manifolds do not have chaotic trajectories, examining the dynamics of families of time one maps of flows gives us a gentler approach to the study of resonance phenomena for diffeomorphisms. Nonetheless, many of the topological features associated with diffeomorphisms cannot occur in time one maps of flows, so additional analysis is required once one has a good intuition for the dynamics of families of flows.

Third, there is a list of codimension one and two bifurcations that occur in generic two parameter families of two dimensional flows. This list can be used to interpret and guide numerical explorations that seek to delineate the dynamics that occur in specific examples.

The interplay between the theoretical analysis of normal forms and interactive computations of phase portraits has played an important role in our developing understanding of families of torus maps. The novel aspect in the analysis of normal forms for our studies has been the necessity of looking carefully at more complicated patterns of saddle connections than have been examined previously.

Recall some of the different types of codimension one and two bifurcations for flows on two dimensional manifolds. In codimension one, there are saddle-node bifurcations at which a pair of equilibrium points coalesce and Hopf bifurcations at which a family of limit cycles ends at an equilibrium. Pairs of limit cycles can coallesce in bifurcations called either double limit cycles or saddle-nodes of periodic orbits. The final type of codimension one bifurcation involves a homoclinic or heteroclinic orbit in which there is a trajectory that terminates at a saddle point as time tends to $\pm\infty$. In codimension two, the list of bifurcations is longer. The local bifurcations involving equilibrium points are cusps, degenerate Hopf bifurcations and the Takens-Bogdanov bifurcations at which an equilibrium has a nilpotent linear part. In addition, there are cusps of limit cycles and various types of bifurcations involving homoclinic and heteroclinic trajectories. For understanding the geometry of the simplest resonance regions occurring in two parameter families, the additional bifurcations that occur involve homoclinic orbits to saddle points with eigenvalues of equal magnitude, saddle-node loops in which there is a homoclinic trajectory for a saddle-node, and double saddle-loops in which a saddle point has two distinct homoclinic trajectories.

Consider a two parameter family of flows on the two dimensional torus with the property that each flow contains at most two equilibrium points. We want to prove as many properties about the bifurcation diagram of such a family as possible.

PROPOSITION. *Let $f : T^2 \to R^2$ be a smooth map with generic singularities and the property that, for each $y \in R^2$, $f^{-1}(y)$ is zero, one or two points. Then the image of f is an annulus with smooth boundaries. The singular set of the map consists of two curves that lie in the same non-trivial homotopy class.*

Proof. The generic singularities of maps between two dimensional manifolds are folds, cusps and the transversal intersection of two fold curves. Since cusps have an adjacent region which is triply covered by the map, cusps do not occur. Nor can there be intersections of two fold curves since each fold has an adjacent region for which there are at least two preimages of each image point. Thus, adjacent to a transversal intersection of two folds is a region in which each point has at least four preimages. We conclude that the singular image of the map is a smooth one dimensional manifold consisting entirely of folds. Thus, the singular image divides the plane into a finite number of regions, one of these regions is the regular image of the map, and the regular image is double covered. Since the Poincaré index of the torus is zero, it follows that there are precisely two boundary components to the image. These components divide the torus into homeomorphic regions. Hence the singular set on the torus must consist of two curves that lie in the same non-trivial homotopy class.

THEOREM. *Let $\dot{x} = \Omega + g(x)$ define a two parameter family of flows on the torus with the property that for each Ω there are at most two equilibrium points of the corresponding flow.*

(1) *The resonance region for the map is an annulus.*

(2) *The flows in the bounded component of the complement of the resonance region (the "hole") do not have a cross-section. In the notation of [1], the flow has type D_ω for a fixed rational ω.*

(3) *There are at least two curves of Hopf bifurcations and six curves of homoclinic bifurcations in the resonance region.*

(4) *The curves of homoclinic bifurcations meet in at least two necklace points at which all of the saddle separatrices of the flow are homoclinic.*

Proof. The first statement is a consequence of the previous proposition since the resonance region is the image of the map $-g : T^2 \rightarrow R^2$. The existence of at least two Takens-Bogdanov points on each boundary of the resonance region was proved in an earlier paper [3]. This implies the existence of at least two curves of Hopf bifurcations and two curves of saddle-node bifurcations. The points on the torus at which the map $-g : T^2 \rightarrow R^2$ has a fold give rise to the saddle-node bifurcations for appropriate parameter values (that depend on the point). Let C be the curve on the torus whose image forms the inner component of the boundary of the image of $-g$. Then C is a smooth closed curve that is homotopically non-trivial on the torus. Observe that if Ω is a parameter value in the hole, then the image of C by the map $-g$ has winding number ± 1 with respect to Ω. Thus, the map $g + \Omega : T^2 \rightarrow R^2$ has image that is an annulus, the origin is in the bounded region of its complement and the image of C has winding number ± 1 with respect to the origin. We assert that this is incompatible with the presence of a cross-section to the flow.

If a flow has a cross-section, then the winding number of the vector field with respect to the origin is zero along any closed curve on the torus. This is readily seen by constructing a homotopy of the vector field to a constant vector field through vector fields without fixed points. Winding vectors do not change under such homotopies. The construction of the homotopy can be accomplished by an initial homotopy that makes the vector field have a constant return time. It is then transverse to a one parameter family of cross-sections. A second homotopy of coordinate changes makes these cross-sections parallel to one another. Finally, a homotopy of the vector field makes it of unit length and orthogonal to the parallel cross-sections. It is then constant. We conclude that $g(x) + \Omega$ has no cross-section.

Consider the vector field associated to a parameter value inside the hole of the resonance region. There are at least two periodic orbits with opposite orientations, but we can say more as well. The torus is divided into annuli by the periodic orbits. Each of these annuli has either trajectories whose orientation does not change or trajectories whose orientation changes by $\pm\pi$. The second type of annulus is called a **Reeb component**. The sum of the changes in orientation from all the annuli is $\pm 2\pi$, corresponding to the index change of the vector field as we traverse the curve along which folds occur in the vector field. If we collapse the annuli in which orientation does not change onto closed curves, we obtain a topological flow on a torus with periodic orbits separated by Reeb components. At least one of the periodic orbits must be attracting (and hence one repelling) since otherwise the sum of changes in orientation from the Reeb components would be 0.

The flow of vector fields for parameters in the hole have annuli that are attracting their boundaries contained in the Reeb components. The presence of an attracting annulus with boundary in Reeb components is a persistent phenomenon under perturbation. Though the size of an attracting annulus may change discontinuously as periodic orbits undergo bifurcation, perturbations of the flow will have an attracting annulus that intersects the original one. As our parameter varies over the disk, we can therefore find continuous families of closed curves that lie in the the Reeb components that contain the boundaries of an attracting annulus.

We next consider the geometry of the vector fields obtained by traversing the parameter curve γ bounding the hole. For each parameter value, there is still a periodic orbit of the homotopy type of those found in the hole since a saddle-node bifurcation will not affect periodic orbits of opposite orientations. Observe that the curve σ traversed on the torus by the saddle-node points lies in a different (non-trivial) homotopy type from the periodic orbits since the winding number of the vector field along this curve is non-zero for parameter values inside the hole. We also assert that the saddle-node points cross Reeb components containing the boundary of an attracting annulus as one traverses γ. If this were not the case, then a continuous family of closed curves in the Reeb components would undergo a non-trivial homotopy along σ. This is a contradiction since these families of closed curves in the Reeb components can be extended continuously over the hole, implying that the homotopy is trivial.

Since the saddle-nodes cross Reeb components as parameters traverse γ, there must be parameter values in γ for which the saddle-node points cross the boundaries of Reeb components. In a generic family of vector fields, these must be codimension two bifurcations which adjoin regions with different numbers of periodic orbits. Examining the list of codimension two bifurcations for families of vector fields with at most one saddle point, the only possibility for these bifurcations are that they are **saddle-node loops**. If the saddle-node has a negative eigenvalue, this means that the unstable separatrix of the saddle-node lies on the boundary of the stable manifold of the saddle-node. Pursuing this logic further, one finds that there must be at least four such bifurcation points corresponding to the boundary components of two distinct Reeb components. ¿From each of these codimension two bifurcations emerges a curve of homoclinic bifurcations in the parameter space. These bifurcation curves are distinct from those that end at the Takens-Bogdanov points since the homoclinic orbits are homotopically non-trivial. We conclude that there are at least six curves of homoclinic bifurcations in our family of vector fields. This ends the proof of the theorem.

REMARKS. The homoclinic bifurcation curves can end only at saddle-node loops in generic families with at most one saddle point. The arguments given above can be extended to show that the homoclinic bifurcation curves that we have identified must cross the resonance region from one boundary component to the other. If one tracks periodic orbits along a homotopically non-trivial closed curve in the resonance region, they follow paths that are homotopic to σ. Thus the saddle-ponts cross "Reeb components" as one traverses the parameter curve. The changes of behavior that occur in the position of saddle points correspond to homoclinic bifurcations. Since the stability of the two boundary components of a Reeb component differ, there must be codimension two bifurcations at which the rotational homoclinic orbits come from a trace zero saddle point. Generically, there are curves of saddle-node bifurcations for periodic orbits that emanate from such codimension two bifurcations. Further analysis of the order with which different homoclinic bifurcations will be encountered on the two boundary components of a Reeb component indicates that the homoclinic bifurcations cross. Thus there are codimension two bifurcation points with double homoclinic cycles. These bifurcations are called **necklace** points in [1].

When one considers the simplest resonance regions for diffeomorphisms rather than flows, the structures described above become more complicated. Curves of homoclinic bifurcations become thickened and the structure of saddle-nodes of periodic orbits becomes more complicated, with the addition of "Chenciner bubbles" where a pair of resonant invariant curves try to approach each other. The reader is referred to [1] for a discussion of what we know about the details of these additional complications that appear in the bifurcation diagrams of two parameter families of diffeomorphisms of the two dimensional torus.

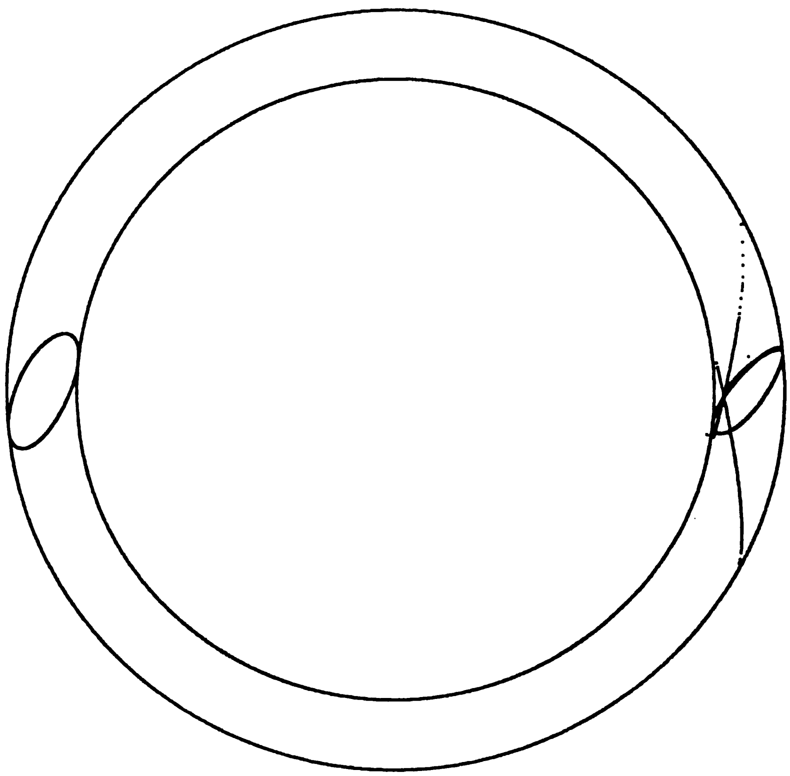

FIGURE 1: A numerically computed simple resonance region for the family of flows

$$\dot{x} = \Omega_x + \cos(2\pi x) + 0.1\cos(2\pi y)$$
$$\dot{y} = \Omega_y + \cos(2\pi x) + 0.1\cos(2\pi y)$$

The entire resonance region with its boundary of saddle-node curves is shown. Half of each small oval is a curve of Hopf bifurcations and half is a curve along which there is a saddle point with determinant one. Curves of parameters with homoclinic bifurcations are shown on one side of the resonance region.

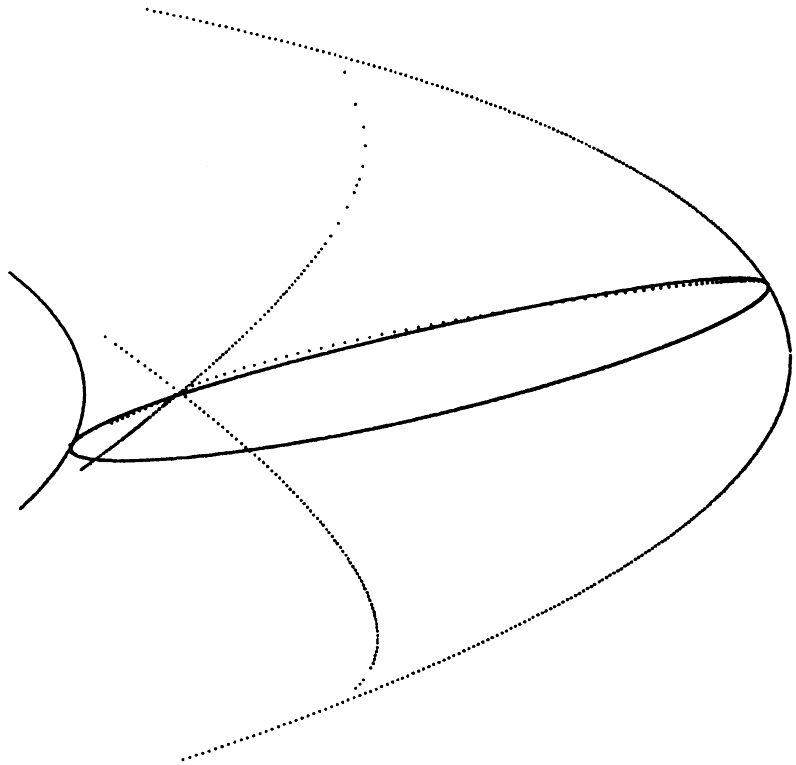

FIGURE 2: An enlarged view of one side of the simple resonance region shown in Figure 1.

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