**SINGULAR SOLUTIONS OF A TRANSMISSION
PROBLEM IN PLANE LINEAR ELASTICITY
FOR WEDGE-SHAPED REGIONS**

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Summary. In plane elasticity, when two different wedge-shaped elastic materials (isotropic, homogeneous) are bonded together along a common edge and subject to tractions on the boundary, the stress field σ will become infinite at the apex. In fact, asymptotically, the displacement u satisfies

$$u \approx \sum_{j=1}^n r^{-i\lambda_j} \sum_{k=0}^{\eta_j-1} \ln^k(r) \Phi_{jk}(\theta) \quad \text{as } r \rightarrow 0,$$

where (r, θ) are the polar coordinates with origin at the wedge apex and the complex numbers λ_j are the roots with positive imaginary part and multiplicity η_j of an explicitly given transcendental equation: $G(\lambda) = 0$.

For applications it is desirable to understand how the number of terms that give rise to unbounded stresses in the above expansion depends on the elasticity constants and on the wedge angles. Setting $N(q) = \sum_{0 < \operatorname{Im}(\lambda_j) < q} \eta_j$, the problem is then to study $N(1)$ as a function of the material constants and wedge angles. A rigorous analysis is presented here for the case in which the wedge angles are equal (symmetric domains). Using the fact that G is an entire function of λ which depends continuously on the parameters of the problem, we develop a “continuity method” which enables us to evaluate the functions $N(\frac{\pi}{a})$ and $N(\frac{\pi}{2a})$ where a is the measure of each wedge angle. This yields bounds on $N(1)$ as well as some information on the zeros of G that correspond to higher order singularities.

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Introduction. Elliptic boundary value problems in domains $\Omega \subset \mathbb{R}^n$ having “singular points” at the boundary have been studied extensively in the literature. A thorough survey can be found in *Kondrat’ev - Oleinik* [15]. In this context the term “singular” is applied to two different kinds of points P :

- (i) $P \in \partial\Omega$ where $\partial\Omega$ is not smooth at P , and
- (ii) $P \in \partial\Omega \cap \Gamma$ where Γ is a surface such that the coefficients of the equation have a jump discontinuity across Γ , e.g. in transmission problems (even if $\partial\Omega$ is smooth near P).

In general, the lack of boundary regularity or the transmission conditions result in singular behavior of the solution near P . Of particular importance in applications is the case where Ω has finitely many *isolated* singular points, which includes points of the type (ii) described above if $\Omega \subset \mathbb{R}^2$. A partial list of references dealing with the problem of isolated singular points includes *Kondrat’ev* [14], *Sovin* [21] and *Grisvard* [10],[11]. In [14], solutions to a single equation in a neighborhood of an n -dimensional conical point were studied using weighted Sobolev spaces. The methods were extended in [21] to include

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transmission problems for systems in two independent variables. A thorough study of two-dimensional problems for one equation is contained in [10], while [11] deals also with some classical equations of mechanics, in the setting of plane polygons.

Due to its importance in industrial and engineering applications, elliptic problems in domains with boundary singularities appear naturally in fracture analysis. One such application arises in semiconductor packaging (see e.g. [9],[12]) where a chip with sharp edges is encapsulated (for protection) in a plastic material. The difference in the elastic properties of the two bonded materials results in the development of cracks (unbounded stresses) within the plastic, undermining its reliability.

A commonly made assumption is that the process is “planar”, with no strain in the x_3 -direction (plane strain). This leads to the study of two-dimensional linear elasticity in a domain $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ like the one in Fig. 0.1, with a jump in the elasticity coefficients across Γ .

Fig. 0.1

Many studies have dealt with finite element approximation in stress analysis near corners, where it becomes infinite. A simple, although expensive, method consists of refining the mesh near the singular point ([7],[10;§8.4.1]). An alternative approach can be found, for example, in [3],[17],[18],[20]; it requires a good knowledge of the asymptotic behavior of the solution near the corner. In particular, one needs to determine the number and form of the singular terms in this expansion as functions of the elastic properties of the materials and the angle at the corner. This paper studies this question in the case described by Fig. 0.1, where we assume that the materials occupying Ω_1 and Ω_2 are isotropic and

homogeneous. We shall also assume, for simplicity, that the domain Ω is “symmetric”, i.e. $a_1 = a_2 = a$.

In §1 we briefly explain how the general theory of elliptic boundary value problems in domains with isolated singular points can be applied to our case, yielding an asymptotic expansion of the solution near the origin; the terms in this expansion are of the form

$$(0.1) \quad cr^{-i\lambda} \ln^k(r) \varphi(\theta) \quad \text{or} \quad cr^m \ln^k(r) \varphi(\theta)$$

where $\ln^k(r) = (\ln(r))^k$, λ, c are constants, k, m nonnegative integers and (r, θ) are the polar coordinates in $\mathbf{R}^2$. Furthermore, the “characteristic exponents” λ will coincide with the solutions, in $\text{Im}(\lambda) > 0$, of an explicitly given transcendental equation

$$(0.2) \quad G(\lambda) = 0.$$

This equation was already derived and, to some extent, numerically studied by *Bogy* [5]; see also [16]. Denoting by $N(q)$ the number of zeros of G (counted with multiplicities) in $0 < \text{Im}(\lambda) < q$, our main objective is to study $N(1)$ (the number of zeros that give rise to unbounded stresses as $r \rightarrow 0$) as a function of the elasticity constants and the angle a . For some particular choices of the parameters, $N(1)$ is easily found. In the general case we use a “continuity method”. The implementation of this method requires auxiliary results on the zeros of Δ , proved in §2. In §3 we obtain a complete description of $N(1)$ in case $a = \frac{\pi}{2}$; in fact, we show that

$$(0.3) \quad N(1) \leq 1 \quad \left(a = \frac{\pi}{2}\right).$$

and give a necessary and sufficient condition (in terms of the elasticity constants) for equality to hold in (0.3).

The transcendental dependence of (0.2) on the angle a makes it difficult to analytically study $N(1)$ as a varies. It turns out to be much easier to work with $N(\frac{\pi}{a})$ when $\frac{\pi}{2} < a < \pi$ and with $N(\frac{\pi}{2a})$ when $0 < a < \frac{\pi}{2}$. This is done in §4 where we show

$$(0.4) \quad 2 \leq N\left(\frac{\pi}{a}\right) \leq 3 \quad \text{if} \quad \frac{\pi}{2} < a \leq \pi$$

and

$$(0.5) \quad N\left(\frac{\pi}{2a}\right) \leq 1 \quad \text{if} \quad 0 < a < \frac{\pi}{2}.$$

We also give a necessary and sufficient condition for $N(\frac{\pi}{a}) = 3$ to hold in (0.4). The results in §4 yield a bound on $N(1)$ and also provide some information on the structure

of higher order singularities, i.e. those terms in (0.1) for which some derivative of the stresses becomes infinite at $r = 0$ (although the stresses themselves remain bounded). The importance of these singularities has been demonstrated by recent numerical techniques for computing the “stress intensity factor” (that is, the constant c in (0.1)); see [4],[22],[23].

The final section of the paper, §5, is devoted to numerical examples and some conclusions.

§1. Behavior near the origin. Let Ω_1, Ω_2 be bounded domains in $\mathbb{R}^2$. We denote by (r, θ) the polar coordinates and assume that

$$\Omega_1 \subset D_1 \equiv \{(r, \theta) : r > 0, 0 < \theta < a\},$$

$$\Omega_2 \subset D_2 \equiv \{(r, \theta) : r > 0, -a < \theta < 0\} \quad (0 < a < \pi)$$

and that $\partial\Omega_i$ consists of three curves: a segment Γ_i on $\{\theta = (-1)^{i-1}a\}$, a curve $\Lambda_i \subset D_i$ and a common segment Γ on $\{\theta = 0\}$.

Consider the system of linear elasticity for isotropic materials in two dimensions,

$$(1.1) \quad \sum_{j=1}^2 \sigma_{ij} = f_i \quad \text{in } \Omega \equiv \Omega_1 \cup \Gamma \cup \Omega_2 \quad (i = 1, 2)$$

where

$$(1.2) \quad \sigma_{11} = \frac{2\mu}{1-n} \left(\frac{\partial u_1}{\partial x_1} + n \frac{\partial u_2}{\partial x_2} \right),$$

$$(1.3) \quad \sigma_{22} = \frac{2\mu}{1-n} \left(n \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right),$$

$$(1.4) \quad \sigma_{12} = \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right).$$

Here,

$$\begin{aligned} \mu &= \text{shear modulus,} \\ n &= \begin{cases} \nu, & \text{for generalized plane stress} \\ \frac{\nu}{1-\nu}, & \text{for plane strain,} \end{cases} \end{aligned}$$

and

$$\nu = \text{Poisson's ratio.}$$

We shall assume that the materials occupying Ω_1 and Ω_2 are homogeneous and isotropic with

$$(1.5) \quad \begin{aligned} \mu &= \mu_i = \text{const.} \geq 0 \quad \text{in } \Omega_i, \\ n &= n_i = \text{const.} \geq 0 \quad \text{in } \Omega_i; \end{aligned}$$

further, the strain energy density is positive definite, which means that

$$(1.6) \quad 0 \leq n \leq 1 \quad (\text{or, equivalently, } 0 \leq \nu \leq \frac{1}{2}).$$

We supplement (1.1) with the following boundary conditions:

$$(1.7) \quad \sigma \cdot N_i = g_i = \begin{pmatrix} g_{i1} \\ g_{i2} \end{pmatrix} \quad \text{on } \Gamma_i \quad (\text{given normal stresses}),$$

where $\sigma = (\sigma_{ij})$, $N_i =$ unit normal vector to Γ_i , and

$$(1.8) \quad u \equiv (u_1, u_2) = 0 \quad \text{on } \Lambda_1 \cup \Lambda_2 \quad (\text{no-displacement}).$$

Finally, for simplicity, we assume that

$$(1.9) \quad f_i, g_i \text{ are smooth, e.g. } C^\infty.$$

From Korn's inequality (see e.g. *Friedrichs* [8]) it follows that there exists a unique solution u of (1.1),(1.7),(1.8) satisfying

$$(1.10) \quad u = (u_1, u_2) \in (W^1(\Omega))^2;$$

here, for any integer $k \geq 0$, $W^k(\Omega)$ denotes the usual Sobolev space of functions defined in Ω with k derivatives in $L^2(\Omega)$. This paper is devoted to the study of the behavior of the solution u in (1.10) near the origin. To find the form of the singular part of the solution we shall apply the results in [14] and [21]. First we need to introduce the weighted Sobolev spaces $W_\alpha^k(\Omega)$.

DEFINITION 1.1 (see [14]). For every nonnegative integer k and $\alpha \in \mathbb{R}$ and any domain $\mathcal{O} \subset \mathbb{R}^2$, denote by $W_\alpha^k(\mathcal{O})$ the space of functions v defined in the domain $\mathcal{O}$ and having a finite norm

$$\|v\|_{k,\alpha,\mathcal{O}}^2 = \sum_{|m| \leq k} \int_{\mathcal{O}} r^{\alpha-2k+2|m|} \left| \frac{\partial^m v}{\partial x^m} \right|^2 dx.$$

If γ is a portion of $\partial\mathcal{O}$, we also define $W_\alpha^{k-\frac{1}{2}}(\gamma)$ as the space of restrictions φ to γ of functions in $W_\alpha^k(\mathcal{O})$ with the norm

$$\|\varphi\|_{k-\frac{1}{2},\alpha,\gamma} = \inf_{\substack{v \in W_\alpha^k(\mathcal{O}) \\ v|_\gamma = \varphi}} \|v\|_{k,\alpha,\mathcal{O}}.$$

The standard theory of elliptic systems (see e.g. *Ciarlet* [6;Chap.6 §3], *Nečas* [19;Chap.5 §3]) together with a scaling argument as in [14] yield the following regularity result

LEMMA 1.2. Let $\delta > 0$ and let $\rho > 0$ be such that $B_\rho(0) \cap \{-a < \theta < a\} \subset \Omega$. Set $u_j^i = u_j|_{\Omega_i}$, where $u = (u_1, u_2)$ is the solution of (1.1)-(1.9) satisfying (1.10). Then

$$(1.11) \quad u_j^i \in W_{2+2\delta}^2(B_\rho(0) \cap \Omega_i).$$

It will be convenient to transform the system (1.1)-(1.9) into a system in a domain $\tilde{\Omega}$ where $\partial\tilde{\Omega}$ near the origin coincides with the line $x_1 = 0$ by setting

$$(1.12) \quad \tilde{u}_j^i(x_1, x_2) = u_j^i(\tilde{x}_1(x_1, x_2), \tilde{x}_2(x_1, x_2)) \quad \text{in } \tilde{\Omega}_i$$

where

$$(1.13) \quad \tilde{x}_1(x_1, x_2) = r \cos\left(\frac{\pi}{2a} \arccos \frac{x_1}{r}\right),$$

$$(1.14) \quad \tilde{x}_2(x_1, x_2) = r \sin\left(\frac{\pi}{2a} \arccos \frac{x_1}{r}\right) \quad (r^2 = x_1^2 + x_2^2),$$

$\tilde{\Omega}_i \subset \tilde{D}_i \equiv \{x_1 > 0, (-1)^{i-1}x_2 > 0\}$ and $\tilde{\Omega} = \tilde{\Omega}_1 \cup \Gamma \cup \tilde{\Omega}_2$. (Observe that the domain $\tilde{\Omega}_i$ is obtained from $\tilde{\Omega}_i$ by “stretching” the angular variable). Notice that, from (1.11),

$$(1.15) \quad \tilde{u}_j^i \in W_{2+2\delta}^2(B_\rho(0) \cap \tilde{\Omega}_i).$$

The above transformation gives an elliptic boundary value problem for the $\tilde{u}_j^i$'s, and, as easily verified,

$$(1.16) \quad \begin{aligned} &\text{the boundary conditions on } \{x_1 = 0\} \\ &\text{as well as the “transmission conditions”} \\ &\text{on } \Gamma \text{ are complementing (in the sense of [1]).} \end{aligned}$$

Thus, we can apply the methods of [21], which we now briefly review. First, after changing variables to polar coordinates and upon localization of the problem near $r = 0$, one obtains a “model problem”, that is, a problem in the union of the two quarter-planes $\tilde{D}_i$. Introducing the independent variable

$$(1.17) \quad t = -\ln(r) \quad , \quad -\infty < t < \infty$$

and reflecting $\tilde{D}_2$ onto $\tilde{D}_1$ we obtain a “duplicated” problem with coefficients independent of t , of the form

$$(1.18) \quad \tilde{L}\left(\theta, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial t}\right)\tilde{w} = \tilde{F}(t, \theta) \quad \text{in } S,$$

$$(1.19) \quad \tilde{B}\left(\theta, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial t}\right)\tilde{w} = \tilde{G}(t, \theta) \quad \text{on } \partial S,$$

where $S = \{(t, \theta) : -\infty < t < \infty, 0 < \theta < \frac{\pi}{2}\}$ and $\tilde{L} = (\tilde{l}_{ij})$, $\tilde{B} = (\tilde{b}_{ij})$, $\tilde{w} = (\tilde{w}_i)$, $\tilde{F} = (\tilde{F}_i)$, $\tilde{G} = (\tilde{G}_i)$, $1 \leq i, j \leq 4$. The problem (1.18),(1.19) can be studied using Fourier transform in the variable t . As in [21] we can derive existence and uniqueness for the model problem in the spaces $W_\alpha^k(\tilde{D}_i)$.

With the aid of lemma 1.2, the methods of [14] (see also [13]) can now be applied to describe the behavior of the solution u in (1.10). For this, let $\mathcal{R}(\lambda)$ denote the resolvent of the system obtained from (1.18),(1.19) by Fourier transformation: $\tilde{w} = \mathcal{R}(\lambda)(\tilde{\varphi}, \tilde{\psi}_0, \tilde{\psi}_1)$ satisfies

$$(1.20) \quad \begin{aligned} \tilde{L}(\theta, \frac{d}{d\theta}, i\lambda)\tilde{w} &= \tilde{\varphi}(\theta), \\ \tilde{B}(0, \frac{d}{d\theta}, i\lambda)\tilde{w} &= \tilde{\psi}_0, \\ \tilde{B}(\frac{\pi}{2}, \frac{d}{d\theta}, i\lambda)\tilde{w} &= \tilde{\psi}_1 \end{aligned}$$

and

$$(1.21) \quad \mathcal{R}(\lambda) : (W^k(I))^4 \times \mathbb{C}^4 \times \mathbb{C}^4 \rightarrow (W^{k+2}(I))^4, \quad I = (0, \frac{\pi}{2}).$$

We recall (see [2]) that $\mathcal{R}(\lambda)$ is a meromorphic function of λ and

$$(1.22) \quad \begin{aligned} &\text{in every strip } |Im(\lambda)| < c_0 \text{ there is} \\ &\text{a number } c = c(c_0) \text{ such that} \\ &\mathcal{R}(\lambda) \text{ has no poles if } |Re(\lambda)| > c. \end{aligned}$$

By [14] [21] (and also [13]) the following theorem holds:

THEOREM 1.3 (Behavior near $r = 0$). *Assume that $\mathcal{R}(\lambda)$ has no poles on the line $Im(\lambda) = 1 + m - \delta$ where $\delta > 0$ and m is a nonnegative integer. Then, if $u = (u_1, u_2)$ is the solution of (1.1)-(1.9) satisfying (1.10),*

$$(1.23) \quad u = u_S + u_R$$

where u_S (the singular part of u) has the form

$$(1.24) \quad u_S = \sum_{j=1}^n r^{-i\lambda_j} \sum_{k=0}^{\eta_j-1} \ln^k(r) \sum_{l=1}^{L_{kj}} a_{jkl} \psi_{jkl}(\theta) + \sum_{\substack{2 \leq k \leq K \\ 0 \leq q \leq Q_k}} r^k \ln^q(r) \varphi_{kq}(\theta) \equiv u_S^0 + u_S^1$$

and u_R (the regular part of u) satisfies

$$(1.25) \quad u_R \in (W^1(\Omega))^2 \cap (W_{2\delta}^{m+2}(B_\rho(0) \cap \Omega_i))^2 \quad (i = 1, 2).$$

In (1.24), $\lambda_1, \dots, \lambda_n$ are the poles of $\mathcal{R}(\lambda)$ in the strip $-\delta < \text{Im}(\lambda) < 1 + m - \delta$ with orders $\eta_1, \dots, \eta_n$, respectively; L_{kj} , K , Q_k are nonnegative integers and the functions ψ_{jkl} , φ_{kq} are infinitely differentiable in $[0, a]$ and in $[-a, 0]$. Furthermore, the functions ψ_{jkl} are independent of u and, if $\text{Im}(\lambda_j) = 0$, then

$$(1.26) \quad a_{jkl} = 0.$$

Notice that the terms in u_S^0 are oscillating unless λ_j is purely imaginary.

REMARK 1.4. The function u_S^1 satisfies (see lemma 3.10 in [14])

$$\begin{aligned} \sum_{j=1}^2 \sigma_{hj}^s - f_h &\in W_{2\delta}^m(\Omega_i) \quad (1 \leq h, i \leq 2), \\ \sigma^s \cdot N_i - g_i &\in \left(W_{2\delta}^{m+\frac{1}{2}}(\Gamma_i)\right)^2 \quad (i = 1, 2), \end{aligned}$$

where (σ_{hj}^s) is defined as (σ_{hj}) in (1.2)-(1.4) but with u replaced by u_S^1 .

REMARK 1.5. Note that, if $\text{Im}(\lambda_j) > 0$, the terms of u_S that give rise to unbounded stresses are of the form

$$r^{-i\lambda_j} \ln^k(r) a_{jkl} \psi_{jkl}(\theta)$$

where $\text{Im}(\lambda_j) \leq 1$.

REMARK 1.6. If the domain is not symmetric, i.e. $a_1 \neq a_2$, then one must replace a by a_i in (1.13), (1.14). In this case it can be shown that (1.16) continues to hold and Theorem 1.3 is still valid.

REMARK 1.7. The arguments that establish Theorem 1.3 apply also to the case where the boundary condition (1.8) is replaced by other boundary conditions; the “characteristic exponents” λ_j are in fact independent of this boundary condition.

We close this section by showing how the problem of finding the characteristic exponents can be reduced to finding the zeros of a particular transcendental equation. Set

$$(1.27) \quad k = \frac{\mu_1}{\mu_2} \quad , \quad m_i = \frac{n_i}{1 + n_i} \quad (i = 1, 2)$$

so that

$$(1.28) \quad 0 \leq m_i \leq \frac{1}{2}.$$

Without loss of generality we shall assume that

$$(1.29) \quad 0 \leq k \leq 1.$$

LEMMA 1.8. A complex number λ_0 is a pole of $\mathcal{R}(\lambda)$ of order q if and only if λ_0 is a zero with multiplicity q of the function

$$\begin{aligned}
(1.30) \quad \Delta(\lambda) &= \Delta(\lambda, a, k, m_1, m_2) \\
&= \left\{ [2(k-1) - 4km_1 + 4m_2] \sinh^2(\lambda a) + 2(k-1) \sin^2(a) \lambda^2 \right\}^2 \\
&\quad + \lambda^2 \sin^2(a) \left\{ [4(k-1) - 4km_1 + 4m_2]^2 \sin^2(a) \right. \\
&\quad \left. + [4(k+1) - 4km_1 - 4m_2]^2 \cos^2(a) \right\} \\
&\quad - [4(k+1) - 4km_1 - 4m_2]^2 \sinh^2(\lambda a) \cosh^2(\lambda a).
\end{aligned}$$

Proof. We begin with a simplification of (1.18),(1.19). This can be obtained by working with the radial and angular displacements:

$$(1.31) \quad \begin{pmatrix} u_r(x) \\ u_\theta(x) \end{pmatrix} = Q(s) \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$

where $x = R(\cos(s), \sin(s))$, $Q(s) = \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix}$. The corresponding $\tilde{u}_r^i, \tilde{u}_\theta^i$ are defined by

$$(1.32) \quad \begin{pmatrix} \tilde{u}_r^i(x) \\ \tilde{u}_\theta^i(x) \end{pmatrix} = \begin{pmatrix} u_r^i(\tilde{x}) \\ u_\theta^i(\tilde{x}) \end{pmatrix} = Q\left(\frac{\pi s}{2a}\right) \begin{pmatrix} u_1^i(\tilde{x}) \\ u_2^i(\tilde{x}) \end{pmatrix} = Q\left(\frac{\pi s}{2a}\right) \begin{pmatrix} \tilde{u}_1^i(x) \\ \tilde{u}_2^i(x) \end{pmatrix}.$$

The system (1.18),(1.19) is then replaced by a system with constant coefficients,

$$(1.33) \quad L\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}\right)w = F \quad \text{in } S,$$

$$(1.34) \quad B\left(\theta, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}\right)w = G \quad \text{on } \partial S,$$

where

$$(1.35) \quad L = L_0 + L_1,$$

$$(1.36) \quad B = B_0 + B_1$$

and

$$(1.37) \quad L_0\left(\frac{2a}{\pi}i\xi, i\lambda\right) = \begin{pmatrix} -\frac{2\lambda^2}{(1-n_1)} - \xi^2 & \frac{(1+n_1)}{(1-n_1)}\xi\lambda & 0 & 0 \\ \frac{(1+n_1)}{(1-n_1)}\xi\lambda & -\lambda^2 - \frac{2\xi^2}{(1-n_1)} & 0 & 0 \\ 0 & 0 & -\frac{2\lambda^2}{(1-n_2)} - \xi^2 & -\frac{(1+n_2)}{(1-n_2)}\xi\lambda \\ 0 & 0 & -\frac{(1+n_2)}{(1-n_2)}\xi\lambda & -\lambda^2 - \frac{2\xi^2}{(1-n_2)} \end{pmatrix},$$

$$(1.38) \quad L_1\left(\frac{2a}{\pi}i\xi, i\lambda\right) = \begin{pmatrix} -\frac{2}{(1-n_1)} & -i\frac{(3-n_1)}{(1-n_1)}\xi & 0 & 0 \\ i\frac{(3-n_1)}{(1-n_1)}\xi & -1 & 0 & 0 \\ 0 & 0 & -\frac{2}{(1-n_2)} & i\frac{(3-n_2)}{(1-n_2)}\xi \\ 0 & 0 & -i\frac{(3-n_2)}{(1-n_2)}\xi & -1 \end{pmatrix},$$

$$(1.39) \quad B_0\left(\frac{\pi}{2}, \frac{2a}{\pi}i\xi, i\lambda\right) = \begin{pmatrix} i\xi & -i\lambda & 0 & 0 \\ \frac{n_1}{(1-n_1)}i\lambda & \frac{i\xi}{(1-n_1)} & 0 & 0 \\ 0 & 0 & -i\xi & -i\lambda \\ 0 & 0 & -\frac{n_2}{(1-n_2)}i\lambda & -\frac{i\xi}{(1-n_2)} \end{pmatrix},$$

$$(1.40) \quad B_1\left(\frac{\pi}{2}, \frac{2a}{\pi}i\xi, i\lambda\right) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ \frac{1}{(1-n_1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{1}{(1-n_2)} & 0 \end{pmatrix},$$

$$(1.41) \quad B_0\left(0, \frac{2a}{\pi}i\xi, i\lambda\right) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ i\xi & -i\lambda & ki\xi & ki\lambda \\ -\frac{n_1}{(1-n_1)}i\lambda & \frac{i\xi}{(1-n_1)} & -\frac{kn_2}{(1-n_2)}i\lambda & \frac{ki\xi}{(1-n_2)} \end{pmatrix}$$

and

$$(1.42) \quad B_1\left(0, \frac{2a}{\pi}i\xi, i\lambda\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & k \\ -\frac{1}{(1-n_1)} & 0 & -\frac{k}{(1-n_2)} & 0 \end{pmatrix}.$$

The Fourier transformed system,

$$\begin{aligned} L\left(\frac{\partial}{\partial\theta}, i\lambda\right)\hat{w} &= \hat{F} \quad \text{in } I, \\ B\left(\theta, \frac{\partial}{\partial\theta}, i\lambda\right)\hat{w} &= \hat{G} \quad \text{on } \partial I, \end{aligned}$$

can be written as a first order system

$$(1.43) \quad \frac{d}{d\theta} U = A(\lambda)U + \Phi \quad \text{in } I,$$

$$(1.44) \quad [C_{11}(\lambda) \quad C_{12}(\lambda)] U\left(\frac{\pi}{2}\right) = \Psi\left(\frac{\pi}{2}\right),$$

$$(1.45) \quad [C_{21}(\lambda) \quad C_{22}(\lambda)] U(0) = \Psi(0),$$

where $U = \begin{pmatrix} \hat{w} \\ \frac{\partial \hat{w}}{\partial \theta} \end{pmatrix}$, $A(\lambda) \in \mathbb{C}^{8 \times 8}$, $C_{ij} \in \mathbb{C}^{4 \times 4}$, $\Phi = \Phi(\theta) \in (W^k(I))^8$, $\Psi(\theta) \in \mathbb{C}^8$ for $\theta = 0, \frac{\pi}{2}$.

Set

$$(1.46) \quad M(\lambda) = \begin{pmatrix} [C_{11}(\lambda) \quad C_{12}(\lambda)] e^{A(\lambda)\frac{\pi}{2}} \\ [C_{21}(\lambda) \quad C_{22}(\lambda)] \end{pmatrix} \in \mathbb{C}^{8 \times 8}.$$

and

$$(1.47) \quad Z(\lambda) = \det[M(\lambda)]$$

From standard ODE theory it follows that λ_0 is a pole of $\mathcal{R}(\lambda)$ of order q if and only if λ_0 is a zero of $Z(\lambda)$ with multiplicity q .

After extensive algebraic calculations*, one concludes that

$$Z(\lambda) = \text{const. } \lambda^2 \Delta(\lambda)$$

where Δ is defined in (1.30), thereby recovering the result in [5] (where (1.30) was derived using Airy functions). $\square$

§2. An auxiliary result on the zeros of Δ . Set

$$(2.1) \quad A = 2(k-1) - 4km_1 + 4m_2,$$

$$(2.2) \quad B = 2(k-1),$$

$$(2.3) \quad C = 4(k+1) - 4km_1 - 4m_2,$$

$$(2.4) \quad E = (A+B)^2 \sin^2(a) + C^2 \cos^2(a).$$

so that

$$(2.5) \quad \Delta(\lambda) = \Delta(\lambda, a, k, m_1, m_2) = \left\{ A \sinh^2(\lambda a) + B \lambda^2 \sin^2(a) \right\}^2 + E \lambda^2 \sin^2(a) - C^2 \sinh^2(\lambda a) \cosh^2(\lambda a).$$

*These computations were checked using the automated algebraic processor MAPLE.

By (1.28),(1.29)

$$(2.6) \quad B \leq 0 \quad , \quad C > 0 \quad ,$$

$$(2.7) \quad (A + B)^2 = 16[k(1 - m_1) - (1 - m_2)]^2 \leq 16[k(1 - m_1) + (1 - m_2)]^2 = C^2$$

$$(2.8) \quad (A - B)^2 = 16(-km_1 + m_2)^2 \leq 16(km_1 + m_2)^2 \leq C^2 .$$

From (2.4),(2.7)

$$(2.9) \quad (A + B)^2 \leq E \leq C^2$$

and from (2.7),(2.8)

$$(2.10) \quad A^2 \leq C^2 \quad , \quad B^2 \leq C^2 .$$

Notice that $\lambda = 0$ is (at least) a double zero of Δ and that $\lambda = i$ is also a zero of Δ (corresponding to the rigid rotation), so that the functions

$$(2.11) \quad F(\lambda) = \frac{\Delta(\lambda)}{\lambda^2} \quad , \quad G(\lambda) = \frac{\Delta(\lambda)}{\lambda - i}$$

are entire.

Our study of the zeros of $\Delta(\lambda)$ will be based upon the fact that these depend continuously on the parameters a, k, m_1, m_2 . Theorem 1.3 suggests that we look for zeros of Δ in strips S_q of the form

$$(2.12) \quad S_q = \{ 0 < \text{Im}(\lambda) < q \} \quad , \quad q > 0.$$

Consider the rectangle

$$(2.13) \quad Q_{q,c} = S_q \cap \{ |\text{Re}(\lambda)| < c \} .$$

If c is sufficiently large, (1.22) implies that the zeros of Δ in S_q actually belong to $Q_{q,c}$. Hence, as the parameters (a, k, m_1, m_2) vary, the zeros of Δ will not leave or enter $Q_{q,c}$ through $|\text{Re}(\lambda)| = c$.

The following lemma shows that the zeros do not leave or enter S_q through $\text{Im}(\lambda) = 0$ and that, in the particular cases $q = \frac{\pi}{a}$ and $q = \frac{\pi}{2a}$ the zeros can move across $\text{Im}(\lambda) = q$ only by going through $\lambda = i\frac{\pi}{a}$ or $\lambda = i\frac{\pi}{2a}$ respectively.

LEMMA 2.1.

- (i) The function $F(\lambda)$ has no zeros on $Im(\lambda) = 0$.
- (ii) If λ_0 is a zero of Δ on $Im(\lambda) = \frac{\pi}{2a}$, $0 < a < \pi$, or on $Im(\lambda) = \frac{\pi}{a}$, $0 < a \leq \pi$, then $Re(\lambda_0) = 0$.

Proof. To prove (i), suppose first that $A \leq 0$. Since

$$(2.14) \quad \sin(a) < a$$

and

$$(2.15) \quad (\lambda a)^2 \leq \sinh^2(\lambda a)$$

for $\lambda \in \mathbf{R}$, $0 < a \leq \pi$, we have

$$(2.16) \quad \begin{aligned} \Delta(\lambda) &\leq (-A - B)^2 \sinh^4(\lambda a) + E \frac{\sin^2(a)}{a^2} \sinh^2(\lambda a) \\ &\quad - C^2 \sinh^2(\lambda a) \cosh^2(\lambda a) \end{aligned}$$

or

$$(2.17) \quad \begin{aligned} \Delta(\lambda) &\leq - (C^2 - (A + B)^2) \sinh^4(\lambda a) \\ &\quad - \left(C^2 - E \frac{\sin^2(a)}{a^2} \right) \sinh^2(\lambda a) \quad , \end{aligned}$$

so that, by (2.9),(2.15),

$$(2.18) \quad F(\lambda) < 0 \quad \text{if } \lambda \in \mathbf{R}, 0 < a \leq \pi.$$

If $A \geq 0$, the use of (2.14),(2.15) implies an inequality like (2.16) but with $(-A - B)^2$ replaced by $(A - B)^2$. Hence, using (2.8) we still get (2.18).

Next we turn to the proof of (ii). Let $\lambda = x + i\frac{\pi}{2a}$ be a zero of Δ ; we have to show that $x = 0$. We can write

$$(2.19) \quad \Delta(x + i\frac{\pi}{2a}) = d_1^R(x) + i d_1^I(x) \quad (x \in \mathbf{R})$$

where

$$(2.20) \quad \begin{aligned} d_1^R &= \left\{ -A \cosh^2(xa) + B \left[x^2 - \left(\frac{\pi}{2a} \right)^2 \right] \sin^2(a) \right\}^2 - 4x^2 \left(\frac{\pi}{2a} \right)^2 B^2 \sin^4(a) \\ &\quad + E \left[x^2 - \left(\frac{\pi}{2a} \right)^2 \right] \sin^2(a) - C^2 \sinh^2(xa) \cosh^2(xa) \end{aligned}$$

and

$$(2.21) \quad d_1^I = \left\{ -A \cosh^2(xa) + B \left[x^2 - \left(\frac{\pi}{2a} \right)^2 \right] \sin^2(a) \right\} 4x \left(\frac{\pi}{2a} \right) B \sin^2(a) \\ + E 2x \left(\frac{\pi}{2a} \right) \sin^2(a).$$

Assuming

$$(2.22) \quad d_1^R(x) = d_1^I(x) = 0 \quad , \quad x \neq 0 \quad ,$$

we shall derive a contradiction. First suppose $k = 1$. Then,

$$B = 0$$

and therefore, from (2.21),(2.22) (since $0 < a < \pi$),

$$E = 0$$

which, upon using (2.4),(2.9), implies that

$$A = 0 \quad \text{and} \quad a = \frac{\pi}{2}.$$

But then,

$$0 = d_1^R(x) = -C^2 \sinh^2(xa) \cosh^2(xa)$$

and hence $x = 0$, a contradiction.

Finally assume $k < 1$ and set

$$(2.23) \quad t = \frac{m_2 - km_1}{1 - k}.$$

Dividing d_1^R by B^2 and d_1^I by $4x \left(\frac{\pi}{2a} \right) B^2 \sin^2(a)$ we get from (2.22):

$$(2.24) \quad 0 = \left\{ -(1 - 2t) \cosh^2(xa) + \left[x^2 - \left(\frac{\pi}{2a} \right)^2 \right] \sin^2(a) \right\}^2 - 4x^2 \left(\frac{\pi}{2a} \right)^2 \sin^4(a) \\ + 4\tilde{E} \left[x^2 - \left(\frac{\pi}{2a} \right)^2 \right] \sin^2(a) - 4\tilde{C}^2 \sinh^2(xa) \cosh^2(xa)$$

and

$$(2.25) \quad 0 = \left\{ -(1 - 2t) \cosh^2(xa) + \left[x^2 - \left(\frac{\pi}{2a} \right)^2 \right] \sin^2(a) \right\} + 2\tilde{E}$$

where

$$(2.26) \quad \tilde{C}^2 = \frac{C^2}{4B^2} = \frac{k+1}{1-k} - t - \frac{2km_1}{1-k}$$

and

$$(2.27) \quad \tilde{E} = \frac{E}{4B^2} = (1-t)^2 \sin^2(a) + \tilde{C}^2 \cos^2(a),$$

since $\frac{(A+B)^2}{4B^2} = (1-t)^2$. Notice that (2.9) implies

$$(2.28) \quad (1-t)^2 \leq \tilde{E} \leq \tilde{C}^2.$$

Now we use (2.25) to replace the first and third summands in (2.24); we obtain, after dividing by 4,

$$\begin{aligned} 0 &= \tilde{E}^2 - x^2 \left(\frac{\pi}{2a} \right)^2 \sin^4(a) + \tilde{E} \left\{ (1-2t) \cosh^2(xa) - 2\tilde{E} \right\} \\ &\quad - \tilde{C}^2 \sinh^2(xa) \cosh^2(xa) \end{aligned}$$

or

$$(2.29) \quad \begin{aligned} 0 &= -\tilde{E}^2 + \tilde{E}(1-2t) \cosh^2(xa) - x^2 \left(\frac{\pi}{2a} \right)^2 \sin^4(a) \\ &\quad - \tilde{C}^2 \sinh^2(xa) \cosh^2(xa). \end{aligned}$$

Since $\tilde{E}^2 \geq \tilde{E}(1-t)^2$ (by (2.28)), (2.29) implies

$$\begin{aligned} 0 &\leq -\tilde{E}(1-2t+t^2) + \tilde{E}(1-2t) \cosh^2(xa) - x^2 \left(\frac{\pi}{2a} \right)^2 \sin^4(a) \\ &\quad - \tilde{C}^2 \sinh^2(xa) \cosh^2(xa) \end{aligned}$$

and, using $\cosh^2(xa) = 1 + \sinh^2(xa)$, we get

$$(2.30) \quad \begin{aligned} 0 &\leq -\tilde{E}t^2 + \tilde{E}(1-2t) \sinh^2(xa) - x^2 \left(\frac{\pi}{2a} \right)^2 \sin^4(a) \\ &\quad - \tilde{C}^2 \sinh^2(xa) - \tilde{C}^2 \sinh^4(xa). \end{aligned}$$

Using the second inequality in (2.28) to replace $\tilde{C}^2$ by $\tilde{E}$ in (2.30), we have

$$(2.31) \quad \begin{aligned} 0 &\leq -\tilde{E} (t^2 + 2t \sinh^2(xa) + \sinh^4(xa)) - x^2 \left(\frac{\pi}{2a} \right)^2 \sin^4(a) \\ &= -\tilde{E} (t + \sinh^2(xa))^2 - x^2 \left(\frac{\pi}{2a} \right)^2 \sin^4(a) \end{aligned}$$

which implies $x = 0$, a contradiction.

The proof for $Im(\lambda) = \frac{\pi}{a}$ is similar. If we set

$$\Delta(x + i\frac{\pi}{a}) = \tilde{d}_1^R(x) + i\tilde{d}_1^I(x) \quad (x \in \mathbb{R})$$

then the formulas for $\tilde{d}_1^R$, $\tilde{d}_1^I$ are like those in (2.20),(2.21) but with

$$-\cosh^2(xa) \quad \text{replaced by} \quad \sinh^2(xa) \quad ,$$

$$-\sinh^2(xa) \quad \text{replaced by} \quad \cosh^2(xa) \quad ,$$

and

$$\frac{\pi}{2a} \quad \text{replaced by} \quad \frac{\pi}{a} .$$

In particular, if $a = \pi$ then $\tilde{d}_1^R(x) = 0$ implies

$$0 = A^2 \sinh^4(xa) - C^2 \sinh^2(xa) \cosh^2(xa) ,$$

so that, by virtue of (2.10),

$$0 \leq -C^2 \sinh^2(xa) (\cosh^2(xa) - \sinh^2(xa)) = -C^2 \sinh^2(xa)$$

which implies $x = 0$.

Thus we may assume $0 < a < \pi$ and proceed as in the proof for $Im(\lambda) = \frac{\pi}{2a}$. Formula (2.30) will be replaced by

$$(2.32) \quad \begin{aligned} 0 \leq & -\tilde{E}t^2 - \tilde{E}(1-2t) \cosh^2(xa) - x^2 \left(\frac{\pi}{a}\right)^2 \sin^4(a) \\ & - \tilde{C}^2 \sinh^2(xa) \cosh^2(xa) \end{aligned}$$

which, using (2.28), yields

$$\begin{aligned} 0 \leq & -\tilde{E}t^2 - \tilde{E}(1-2t) \cosh^2(xa) - x^2 \left(\frac{\pi}{a}\right)^2 \sin^4(a) \\ & - \tilde{E} \cosh^4(xa) + \tilde{E} \cosh^2(xa) \end{aligned}$$

or

$$0 \leq -\tilde{E} (t^2 - 2t \cosh^2(xa) + \cosh^4(xa)) - x^2 \left(\frac{\pi}{a}\right)^2 \sin^4(a)$$

and this again implies $x = 0$. $\square$

From lemma 2.1 we see that, in order to study the dependence of the zeros λ of Δ in $S_{\frac{\pi}{2a}}$, $S_{\frac{\pi}{a}}$ on the parameters, it will be useful to compute the first few terms in the power series expansion of $\Delta(\lambda)$ about $\lambda = i\frac{\pi}{2a}$ and about $\lambda = i\frac{\pi}{a}$. To do this, we evaluate

$$\begin{aligned}
 d_2(x) \equiv \Delta\left(i\left(\frac{\pi}{2a} - x\right)\right) &= \left\{ A \cos^2(xa) + B \left(x - \frac{\pi}{2a}\right)^2 \sin^2(a) \right\}^2 \\
 (2.33) \quad &- E \left(x - \frac{\pi}{2a}\right)^2 \sin^2(a) + C^2 \sin^2(xa) \cos^2(xa) \\
 &= a_{20} + a_{21}x + a_{22}x^2 + \dots,
 \end{aligned}$$

and

$$\begin{aligned}
 d_3(x) \equiv \Delta\left(i\left(\frac{\pi}{a} - x\right)\right) &= \left\{ A \sin^2(xa) + B \left(x - \frac{\pi}{a}\right)^2 \sin^2(a) \right\}^2 \\
 (2.34) \quad &- E \left(x - \frac{\pi}{a}\right)^2 \sin^2(a) + C^2 \sin^2(xa) \cos^2(xa) \\
 &= a_{30} + a_{31}x + a_{32}x^2 + \dots
 \end{aligned}$$

where

$$\begin{aligned}
 a_{20} &= (A + Bf^2)^2 - Ef^2, \\
 a_{21} &= \frac{\pi}{a} \sin^2(a) [-2(A + Bf^2)B + E], \\
 (2.35) \quad a_{22} &= 4B^2f^2 \sin^2(a) + 2(A + Bf^2)(B \sin^2(a) - Aa^2) \\
 &\quad - E \sin^2(a) + C^2a,
 \end{aligned}$$

$$\begin{aligned}
 a_{30} &= g^2 (B^2g^2 - E), \\
 a_{31} &= \frac{2\pi}{a} \sin^2(a) (-2B^2g^2 + E), \\
 (2.36) \quad a_{32} &= 4B^2g^2 \sin^2(a) + 2Bg^2 (Aa^2 + B \sin^2(a)) \\
 &\quad - E \sin^2(a) + C^2a,
 \end{aligned}$$

and

$$(2.37) \quad f = f(a) = \frac{\pi \sin(a)}{2a}, \quad g = g(a) = \frac{\pi \sin(a)}{a}.$$

Finally, to study the purely imaginary zeros of Δ near $\lambda = i$, we shall also need

$$(2.38) \quad d_4(x) \equiv iG(ix) = a_{40} + a_{41}(x - 1) + \dots$$

in which

$$\begin{aligned}
a_{40} &= 2 \sin(a) \left[(A+B) 2 \sin^2(a) (Aa \cos(a) + B \sin(a)) \right. \\
&\quad \left. - E \sin(a) - C^2 a \cos(a) (\sin^2(a) - \cos^2(a)) \right] , \\
(2.39) \quad a_{41} &= 2 \left[Aa^2 (\cos^2(a) - \sin^2(a)) + B \sin^2(a) \right] (A+B) \sin^2(a) \\
&\quad + (2Aa \sin(a) \cos(a) + 2B \sin^2(a))^2 - E \sin^2(a) \\
&\quad + C^2 \left[-4a^2 \sin^2(a) \cos^2(a) + a^2 (\cos^2(a) - \sin^2(a))^2 \right] .
\end{aligned}$$

Formulas (2.33)-(2.39) will be used in the next two sections.

§3. The zeros of Δ . Case $a = \frac{\pi}{2}$.

DEFINITION 3.1. For $q > 0, 0 < a \leq \pi$, define

$$(3.1) \quad N(q) = N(q)(a, k, m_1, m_2) = \sum_{j=1}^n \eta_j ,$$

where $\lambda_1, \dots, \lambda_n$ are the zeros of $G(\lambda) = \frac{\Delta(\lambda)}{\lambda - i}$ in S_q , with multiplicities $\eta_1, \dots, \eta_n$, respectively.

Notice that in the above definition S_q can be replaced by $Q_{q,c}$ where c is given by (1.22).

THEOREM 3.2. Let $a = \frac{\pi}{2}$. Then, under the hypotheses (1.28),(1.29)

$$(3.2) \quad N(1)\left(\frac{\pi}{2}, k, m_1, m_2\right) = \begin{cases} 1 & \text{if } m_2 - (1-k) < km_1 < m_2, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 3.3. From the definition of Δ it is clear that if λ is a zero, then so is $(-\bar{\lambda})$. Thus, if there exists exactly one zero in a strip S_q (i.e. $N(q) = 1$), this zero must be purely imaginary.

If $a = \frac{\pi}{2}$, formulas (2.33),(2.35) take the form

$$(3.3) \quad d_2(x) = \Delta(i(1-x)) = a_{21}x + a_{22}x^2 + \dots$$

and

$$\begin{aligned}
(3.4) \quad a_{21} &= 2(A+B)(A-B) , \\
a_{22} &= 4B^2 + 2(A+B) \left[B - A \left(\frac{\pi}{2} \right)^2 \right] - (A+B)^2 + C^2 \left(\frac{\pi}{2} \right)^2 .
\end{aligned}$$

We shall abbreviate

$$\begin{aligned}\Delta(\lambda) &= \Delta(\lambda, k, m_1, m_2) = \Delta(\lambda, \frac{\pi}{2}, k, m_1, m_2) \quad , \\ G(\lambda) &= G(\lambda, k, m_1, m_2) = G(\lambda, \frac{\pi}{2}, k, m_1, m_2) \quad , \dots\end{aligned}$$

and we also define

$$(3.5) \quad \hat{d}_2(x) \equiv \frac{d_2(x)}{x} = -iG(i(1-x))$$

so that

$$(3.6) \quad \hat{d}_2(x) = a_{21} + a_{22}x + \dots$$

To prove Theorem 3.2 we need two lemmas.

LEMMA 3.4 (Case $m_1 = m_2 = 0$; $a = \frac{\pi}{2}$).

- (i) *The only zero of $G(\lambda, 1, 0, 0)$ in $\overline{S_1}$ is $\lambda = i$, and it is a simple zero.*
- (ii) *The only zero of $G(\lambda, k, 0, 0)$ ($0 \leq k \leq 1$) on $Im(\lambda) = 1$ is $\lambda = i$, and it is a simple zero.*

Proof. The proof of (i) is immediate since

$$\Delta(\lambda, 1, 0, 0) = -16 \sinh^2(\pi\lambda).$$

From lemma 2.1 we know that the only possible zero of Δ on $Im(\lambda) = 1$ is at $\lambda = i$. If $m_1 = m_2 = 0$, then

$$A = B = 2(k-1) \quad , \quad C = 4(k+1)$$

so that

$$a_{21} = 0 \quad , \quad a_{22} = 4B^2 \left(1 - \frac{\pi^2}{4}\right) + C^2 \frac{\pi^2}{4} > 0$$

and therefore, by (3.6), we conclude that $\lambda = i$ is a simple zero of G , thereby proving (ii). $\square$

LEMMA 3.5 (“Transitions of $N(1)$ ”; $a = \frac{\pi}{2}$).

- (i) $G(i, k, m_1, m_2) = 0$ if and only if

$$(3.7) \quad km_1 = m_2 \quad \text{or} \quad km_1 = m_2 - (1-k)$$

(ii) Let $k^0 m_1^0 = m_2^0$, $0 \leq k^0 < 1$. Then, there exists $\epsilon^0 = \epsilon^0(k^0, m_1^0, m_2^0) > 0$ such that if $\|(k, m_1, m_2) - (k^0, m_1^0, m_2^0)\| < \epsilon^0$

$$(3.8) \quad N(1)\left(\frac{\pi}{2}, k, m_1, m_2\right) = \begin{cases} N(1)\left(\frac{\pi}{2}, k^0, m_1^0, m_2^0\right) & \text{if } km_1 \geq m_2, \\ N(1)\left(\frac{\pi}{2}, k^0, m_1^0, m_2^0\right) + 1 & \text{if } km_1 < m_2. \end{cases}$$

(iii) Let $k^1 m_1^1 = m_2^1$, $0 \leq k^1 < 1$. Then, there exists $\epsilon^1 = \epsilon^1(k^1, m_1^1, m_2^1) > 0$ such that if $\|(k, m_1, m_2) - (k^1, m_1^1, m_2^1)\| < \epsilon^1$

$$(3.9) \quad N(1)\left(\frac{\pi}{2}, k, m_1, m_2\right) = \begin{cases} N(1)\left(\frac{\pi}{2}, k^1, m_1^1, m_2^1\right) & \text{if } km_1 \leq m_2 - (1 - k), \\ N(1)\left(\frac{\pi}{2}, k^1, m_1^1, m_2^1\right) + 1 & \text{if } km_1 > m_2 - (1 - k). \end{cases}$$

Proof. The proof of (i) is immediate from (3.4),(3.6). Indeed, $a_{21} = 0$ if and only if $A = \pm B$, and this condition is equivalent to (3.7).

Now recall that, as the parameters vary, the zeros of G cannot enter or leave $Q_{1,c}$ through $Im(\lambda) = 0$. Hence, using the continuity of the zeros as functions of (k, m_1, m_2) , we see that to prove (ii) it suffices to show that the zero which is at $\lambda = i$ for $(k, m_1, m_2) = (k^0, m_1^0, m_2^0)$ lies outside $Q_{1,c}$ for $km_1 > m_2$ and inside $Q_{1,c}$ for $km_1 < m_2$, if (k, m_1, m_2) is sufficiently near (k^0, m_1^0, m_2^0) . Similarly, (iii) will follow once we prove that, for (k, m_1, m_2) close to (k^1, m_1^1, m_2^1) , the zero which is at $\lambda = i$ when $(k, m_1, m_2) = (k^1, m_1^1, m_2^1)$ moves outside $Q_{1,c}$ if $km_1 < m_2 - (1 - k)$ and inside if $km_1 > m_2 - (1 - k)$.

But from (3.4),(3.6)

$$(3.10) \quad \begin{aligned} \left. \frac{\partial}{\partial x} \hat{d}_2 \right|_{k m_1 = m_2}^{x=0} &= a_{22} \Big|_{k m_1 = m_2} = 4B^2 \left(1 - \frac{\pi^2}{4}\right) + C^2 \frac{\pi^2}{4} \Big|_{k m_1 = m_2} \\ &= 4\pi^2 \left\{ [1 + k(1 - 2m_1)]^2 - [1 - k]^2 \right\} + 16(1 - k)^2 > 0 \end{aligned}$$

and therefore, by the Implicit Function Theorem, the zero of G which is at $\lambda = i$ when $(k, m_1, m_2) = (k^0, m_1^0, k^0 m_1^0)$ is of the form

$$(3.11) \quad \lambda = i(1 - x).$$

Furthermore, in (3.11) a first order approximation (in $(k - k^0), (m_1 - m_1^0), (m_2 - m_2^0)$) is given by

$$(3.12) \quad x \approx - \frac{a_{21}}{a_{22} \Big|_{k m_1 = m_2}}.$$

Since,

$$(3.13) \quad a_{21} > 0 \quad \text{for} \quad (-B \geq) B > A$$

and $a_{21} < 0$ for $-B > A > B$ (here we use $k < 1$), we conclude, from (3.10),(3.12),

$$(3.14) \quad \begin{aligned} x &< 0 & \text{for} & \quad km_1 > m_2, \\ x &> 0 & \text{for} & \quad km_1 < m_2, \end{aligned}$$

provided (k, m_1, m_2) is sufficiently close to (k^0, m_1^0, m_2^0) . Now (3.8) follows from (3.11) and (3.14).

The proof of (3.9) is similar. Instead of (3.10) we use

$$(3.15) \quad \left. \frac{\partial}{\partial x} \hat{d}_2 \right|_{km_1 = m_2 - (1-k)}^{x=0} = a_{22} \Big|_{km_1 = m_2 - (1-k)} = 4B^2 + C^2 \frac{\pi^2}{4} > 0,$$

while (3.12),(3.13) are replaced by

$$(3.16) \quad x \approx - \frac{a_{21}}{a_{22} \Big|_{km_1 = m_2 - (1-k)}}.$$

and

$$(3.17) \quad a_{21} > 0 \quad \text{for} \quad A > -B (\geq B).$$

In this case, we conclude

$$\begin{aligned} x &< 0 & \text{for} & \quad km_1 < m_2 - (1-k), \\ x &> 0 & \text{for} & \quad km_1 > m_2 - (1-k) \end{aligned}$$

(if (k, m_1, m_2) is near (k^1, m_1^1, m_2^1)) which, together with (3.12) yields (3.10). $\square$

Proof of Theorem 3.2. Fix $k, 0 \leq k < 1$. From (1.22) and lemmas 2.1,3.5 we conclude that there exists a nonnegative integer $M(k)$ such that

$$(3.18) \quad N(1)(\frac{\pi}{2}, k, m_1, m_2) = \begin{cases} M(k) + 1 & \text{if } m_2 - (1-k) < km_1 < m_2, \\ M(k) & \text{otherwise.} \end{cases}$$

To find $M(k)$ we use lemma 3.4: part (ii) implies that $\frac{G(\lambda, k, 0, 0)}{\lambda - i}$ is analytic and (by (1.22) and lemma 2.1) it never vanishes on $\partial Q_{1,c}$; finally, by part (i) and the continuity of

the zeros as functions of the parameters, we conclude that $\frac{G(\lambda, k, 0, 0)}{\lambda - i}$, or, equivalently, $G(\lambda, k, 0, 0)$ has no zeros in $Q_{1,c}$. Thus

$$M(k) \equiv 0 \quad , \quad 0 \leq k < 1.$$

It remains to consider the case $k = 1$. Let $m_2 > m_1$ and let $0 \leq k_0 < 1$ be such that

$$(3.19) \quad m_2 - (1 - k) > km_1 \quad \text{for} \quad k \in I_0 \equiv [k_0, 1) \quad ,$$

so that, for any $k \in I_0$,

$$(3.20) \quad N(1)\left(\frac{\pi}{2}, k, m_1, m_2\right) = 0.$$

Again, applying (1.22) and lemma 2.1 together with (3.19) and lemma 3.5 we deduce that no zeros of G can enter or exit S_1 as $k \rightarrow 1$. Hence, by (3.20),

$$(3.21) \quad N(1)\left(\frac{\pi}{2}, 1, m_1, m_2\right) = 0 \quad \text{if} \quad m_2 > m_1 \quad ,$$

and, since $\Delta(\lambda, 1, m_1, m_2) = \Delta(\lambda, 1, m_2, m_1)$, (3.21) also holds for $m_1 \geq m_2$. $\square$

REMARK 3.6. The method used in this section can also be applied, with minor changes, to estimate $N(m)\left(\frac{\pi}{2}, k, m_1, m_2\right)$, for integer $m > 1$; for example:

except in the trivial case $k = 1, m_1 = m_2$ (where $N(2) = 1$) we have

$$(3.22) \quad 2 \leq N(2)\left(\frac{\pi}{2}, k, m_1, m_2\right) \leq 3.$$

Furthermore, $N(2) = 3$ if and only if

$$(3.23) \quad m_2 - 2(1 - k) < km_1 < m_2.$$

§4. The zeros of Δ . General case: $0 < a \leq \pi$. Formula (1.30) shows that, in contrast with the algebraic dependence on k, m_1, m_2 , Δ depends on the angle a in a transcendental way. Thus, the study of $N(1)$ as a varies becomes much more complicated. However, due to lemma 2.1, a rigorous study is possible if we replace $N(1)$ by $N\left(\frac{\pi}{a}\right)$ or $N\left(\frac{\pi}{2a}\right)$. Indeed, in this section we prove:

THEOREM 4.1 (Case $\frac{\pi}{2} < a \leq \pi$). Fix k, m_1, m_2 satisfying (1.28), (1.29), and let $\frac{\pi}{2} < a \leq \pi$.

(i) If $m_2 - 2(1 - k) < km_1 < m_2$, then there exists $a_0 = a_0(k, m_1, m_2)$ such that

$$(4.1) \quad N\left(\frac{\pi}{a}\right)(a, k, m_1, m_2) = \begin{cases} 3 & \text{if } \frac{\pi}{2} < a < a_0 < \pi, \\ 2 & \text{if } a_0 \leq a \leq \pi. \end{cases}$$

(ii) If $km_1 \geq m_2$ or $km_1 \leq m_2 - 2(1 - k)$, then

$$(4.2) \quad N\left(\frac{\pi}{a}\right)(a, k, m_1, m_2) = 2 \quad , \quad \frac{\pi}{2} < a \leq \pi.$$

Notice that if $a = \frac{\pi}{2}$, $N\left(\frac{\pi}{a}\right) = N(2)$ is described in Remark 3.6.

THEOREM 4.2 (Case $0 < a < \frac{\pi}{2}$). Fix k, m_1, m_2 satisfying (1.28), (1.29). Then

$$(4.3) \quad N\left(\frac{\pi}{2a}\right)(a, k, m_1, m_2) \leq 1 \quad , \quad 0 < a \leq \frac{\pi}{2}.$$

We shall first prove Theorem 4.1. To this end we shall apply the method of continuity to include also variations in the angle a while k, m_1, m_2 remain fixed. Remark 3.6 provides us with the necessary information to use $a = \frac{\pi}{2}$ as our starting point. However, the analysis is easier if we start from $a = \pi$:

LEMMA 4.3 (Case $a = \pi$). Let $a = \pi$. Then the only zeros of G in $\overline{S_1}$ are

$$\lambda = \pm \frac{1}{2\pi} \ln \left(\frac{A + C}{C - A} \right) + \frac{1}{2}i \quad (\text{simple if } A \neq 0, \text{ double if } A = 0)$$

and $\lambda = i$ (simple).

In particular, $N(1)(\pi, k, m_1, m_2) \equiv 2$.

Proof. The proof is obvious since

$$\Delta(\lambda, \pi, k, m_1, m_2) = A^2 \sinh^4(\lambda\pi) - C^2 \sinh^2(\lambda\pi) \cosh^2(\lambda\pi). \quad \square$$

The following two lemmas will also be needed to prove Theorem 4.1.

LEMMA 4.4 (Case a near π). *There exists an $\epsilon = \epsilon(k, m_1, m_2) > 0$ such that*

$$(4.4) \quad N\left(\frac{\pi}{a}\right)(a, k, m_1, m_2) = 2 \quad \text{if } \pi - \epsilon < a \leq \pi.$$

Proof. Using the continuity of the zeros and lemma 4.3, we see that, in order to prove (4.4), it suffices to show that the zero of G which is at $\lambda = i$ when $a = \pi$ “leaves” $S_{\frac{\pi}{a}}$ as a decreases. Thus, it is enough to show that there exists a function $x = x(a)$ defined for $|a - \pi| < \epsilon$ (for some small $\epsilon > 0$) satisfying

$$(4.5) \quad \begin{aligned} d_4(x(a), a, k, m_1, m_2) &\equiv 0 \quad , \quad x(\pi) = 1 \\ \text{and } x(a) &> \frac{\pi}{a} \quad \text{for } \pi - \epsilon < a < \pi. \end{aligned}$$

(Recall d_4 was defined in (2.38)). But, from (2.38),(2.39),

$$\frac{\partial}{\partial x} d_4 \Big|_{\substack{x=1 \\ a=\pi}} = a_{41} \Big|_{a=\pi} = \pi^2 C^2$$

and

$$\frac{\partial}{\partial a} d_4 \Big|_{\substack{x=1 \\ a=\pi}} = \frac{\partial}{\partial a} a_{40} \Big|_{a=\pi} = (-2)(-\pi)C^2 = 2\pi C^2.$$

Hence, (4.5) follows from the Implicit Function Theorem since

$$\frac{\partial}{\partial a} \left(\frac{\pi}{a} \right) \Big|_{a=\pi} = -\frac{1}{\pi} > -\frac{2}{\pi} = x'(\pi). \quad \square$$

LEMMA 4.5 (“Transitions of $N(\frac{\pi}{a})$ ”; $\frac{\pi}{2} < a < \pi$). *Fix k, m_1, m_2 . Then,*

(i) *the equation*

$$(4.6) \quad \Delta\left(i\frac{\pi}{a}, a, k, m_1, m_2\right) = 0 \quad , \quad \frac{\pi}{2} < a < \pi$$

has at most one solution. It has exactly one solution if and only if

$$(4.7) \quad m_2 - 2(1 - k) < km_1 < m_2.$$

(ii) *Suppose that (4.7) holds and let a_0 be the solution of (4.6). Then there exists an $\epsilon = \epsilon(k, m_1, m_2) > 0$ such that*

$$(4.8) \quad N\left(\frac{\pi}{a}\right)(a, k, m_1, m_2) = N\left(\frac{\pi}{a_0}\right)(a_0, k, m_1, m_2) + 1 \quad \text{if } a_0 - \epsilon < a < a_0.$$

Proof. From (2.34),(2.36) we see that (4.6) is equivalent to

$$a_{30} = 0 \quad , \quad \text{that is (since } g(a) \neq 0 \text{ for } \frac{\pi}{2} < a < \pi)$$

$$(4.9) \quad \left(\frac{\pi \sin(a)}{a} \right)^2 B^2 - \left[(A+B)^2 \sin^2(a) + C^2 \cos^2(a) \right] = 0 \quad , \quad \frac{\pi}{2} < a < \pi .$$

Replacing $\cos^2(a)$ by $(1 - \sin^2(a))$ in (4.9) we see that (4.9) is equivalent to

$$\sin^2(a) = h(a)^2 \quad , \quad \frac{\pi}{2} < a < \pi ,$$

or

$$(4.10) \quad \sin(a) = h(a) \quad , \quad \frac{\pi}{2} < a < \pi ,$$

where

$$(4.11) \quad h(a) = \frac{Ca}{\left\{ B^2 \pi^2 + \left[C^2 - (A+B)^2 \right] a^2 \right\}^{\frac{1}{2}}}$$

(Recall $C \geq 0$, $C^2 \geq (A+B)^2$).

But

$$(4.11) \quad h'(a) = \frac{CB^2\pi^2}{\left\{ B^2 \pi^2 + \left[C^2 - (A+B)^2 \right] a^2 \right\}^{\frac{3}{2}}} \geq 0$$

and therefore (4.10) admits at most one solution, and precisely one if and only if

$$h\left(\frac{\pi}{2}\right) < 1 ,$$

that is, if

$$\frac{C}{\left\{ 4B^2 + C^2 - (A+B)^2 \right\}^{\frac{1}{2}}} < 1$$

or

$$(4.12) \quad 0 < 4B^2 - (A+B)^2 = (B-A)(3B+A) .$$

Finally, since $B \leq 0$, (4.12) is equivalent to

$$-3B > A > B$$

which (due to (2.1),(2.2)) is the same as (4.7); hence, (i) is proved.

To prove (ii) we use, once again, (1.22), lemma 2.1 and the continuity of the zeros. In this way, (ii) reduces to showing that the zero which is at $\lambda = i\frac{\pi}{a_0}$ when $a = a_0$ is “entering” $S_{\frac{\pi}{2a}}$ as a decreases. In other words, we must show that there exists a function $x = x(a)$ defined for $|a - a_0| < \epsilon$ satisfying (see (2.34),(2.36))

$$(4.13) \quad \begin{aligned} d_3(x(a), a, k, m_1, m_2) &\equiv 0 \quad , \quad x(a_0) = 0 \\ \text{and } x(a) &> 0 \quad \text{if } a_0 - \epsilon < a < a_0 . \end{aligned}$$

But, from (2.34),(2.36),

$$(4.14) \quad \begin{aligned} B^2 g^2 - E \Big|_{a=a_0} &= 0 \quad , \\ \frac{\partial}{\partial x} d_3 \Big|_{\substack{x=0 \\ a=a_0}} &= \frac{2\pi}{a} \sin^2(a) (-2B^2 g^2 + E) \Big|_{a=a_0} \quad , \text{ and} \\ \frac{\partial}{\partial a} d_3 \Big|_{\substack{x=0 \\ a=a_0}} &= \frac{\partial}{\partial a} [g^2 (B^2 g^2 - E)] \Big|_{a=a_0} . \end{aligned}$$

Hence, using (4.14),

$$\frac{\partial}{\partial x} d_3 \Big|_{\substack{x=0 \\ a=a_0}} = -\frac{2\pi}{a_0} \sin^2(a_0) B^2 g(a_0)^2 < 0$$

and

$$\frac{\partial}{\partial a} d_3 \Big|_{\substack{x=0 \\ a=a_0}} = g(a_0)^2 \left(-2B^2 \frac{g(a_0)^2}{a_0} + 2C^2 \frac{\cos(a_0)}{\sin(a_0)} \right) < 0 .$$

An application of the Implicit Function Theorem now gives (4.13). $\square$

Proof of Theorem 4.1. The proof is an easy consequence of lemmas 4.3-4.5 and the continuity of the zeros of G in the parameter a . Indeed, if we fix k, m_1, m_2 and start decreasing a from π , lemma 4.4 implies that, for a while,

$$(4.15) \quad N\left(\frac{\pi}{a}\right)(a, k, m_1, m_2) = 2 .$$

By (1.22) and lemma 2.1 we know that the only way (4.15) will cease to hold as we continue decreasing a , is for a zero of G to enter or leave $S_{\frac{\pi}{2a}}$ through $\lambda = i\frac{\pi}{a}$. But, by lemma 4.5, this can happen at most once. Furthermore, it can only happen if $m_2 - 2(1 - k) < km_1 < m_2$, and in this case (by (4.8)) the zero is entering $S_{\frac{\pi}{2a}}$. $\square$

The proof of Theorem 4.2 is again based on the continuity of the zeros of G . However, in this case we shall fix the angle a , $0 < a < \frac{\pi}{2}$, and vary the parameters k, m_1, m_2 . The key steps in the continuity method are established in several lemmas. Lemma 4.6 will help us to discover whether a zero of Δ on $Im(\lambda) = \frac{\pi}{2a}$ is entering or leaving the region $S_{\frac{\pi}{2a}}$. The remaining lemmas discuss the behavior of $N(\frac{\pi}{2a})$ in increasing generality on the range of the parameters.

LEMMA 4.6. Let $0 < a < \frac{\pi}{2}$ and assume that

$$(4.16) \quad \Delta(i\frac{\pi}{2a}, a, k, m_1, m_2) = 0.$$

Then

$$(4.17) \quad \frac{\partial}{\partial x} d_2(0, a, k, m_1, m_2) < 0$$

(d_2 was defined in (2.33)).

In particular, the zero at $\lambda = i\frac{\pi}{2a}$ is simple.

Proof. By (2.33),(2.35), the relation (4.16) is equivalent to

$$(4.18) \quad (A + Bf^2)^2 - f^2 E = 0$$

and (4.17) is equivalent to

$$(4.19) \quad -2(A + Bf^2)B + E < 0.$$

Using (4.18) to replace E in (4.19), we get

$$-2(A + Bf^2)B + \frac{1}{f^2}(A + Bf^2)^2 < 0$$

or

$$(4.20) \quad (A + Bf^2)(A - Bf^2) < 0.$$

Suppose that

$$(A + Bf^2) \geq 0.$$

Then, since $B \leq 0$ and $f \geq 1$,

$$(A + Bf^2)^2 \leq (A + B)^2$$

and therefore, since $E > (A + B)^2$,

$$0 = (A + Bf^2)^2 - f^2 E < (A + B)^2 - f^2 (A + B)^2 \leq 0,$$

a contradiction. Thus, we must have

$$(4.21) \quad (A + Bf^2) < 0.$$

We next prove that

$$(4.22) \quad (A - Bf^2) > 0.$$

Indeed, assume that

$$(A - Bf^2) \leq 0$$

so that

$$(4.23) \quad A \leq Bf^2 \leq Bf \leq B \leq 0.$$

Then, from (4.18),

$$(4.24) \quad 0 < (A + Bf^2)^2 - f^2(A + B)^2 = (A + Bf^2 - fA - fB)(A + Bf^2 + fA + fB).$$

But, by (4.23),

$$\begin{aligned} (A + Bf^2 - fA - fB) &= (Bf - A)(f - 1) \geq 0, \\ (A + Bf^2 + fA + fB) &\leq 0, \end{aligned}$$

and this contradicts (4.24).

Having proved (4.21) and (4.22), the assertion (4.20) follows. $\square$

LEMMA 4.7 (Case $m_1 = m_2 = 0$; $0 < a < \frac{\pi}{2}$). *Let $0 < a < \frac{\pi}{2}$. Then*

- (i) $N(\frac{\pi}{2a})(a, 1, 0, 0) = 0$.
- (ii) *There exists a unique $k_0 = k_0(a)$ such that*

$$(4.25) \quad \Delta(i\frac{\pi}{2a}, a, k_0, 0, 0) = 0.$$

Furthermore, $0 < k_0 < 1$.

- (iii) *There exists $\epsilon = \epsilon(a) > 0$ such that*

$$N(\frac{\pi}{2a})(a, k, 0, 0) = 1 \quad \text{for} \quad k_0 - \epsilon < k < k_0.$$

Proof. From (2.1)-(2.5),

$$(4.26) \quad \Delta(\lambda, a, 1, 0, 0) = 64 [\lambda^2 \sin^2(a) \cos^2(a) - \sinh^2(\lambda a) \cosh^2(\lambda a)].$$

Then, using the inequalities

$$\cosh(z) \geq \frac{\sinh(z)}{z} \quad (z \in \mathbf{R}) \quad , \quad \frac{\sin(z)}{z} \geq \cos(z) \quad (0 \leq z \leq \pi) \quad ,$$

it is easy to see that the only zeros of Δ in $\overline{S_{\frac{\pi}{2a}}}$ are $\lambda = 0$ (double) and $\lambda = i$ (simple), thereby proving (i).

Using (2.33),(2.35) and (2.1)-(2.4), (4.25) is equivalent to the equation

$$(4.27) \quad 0 = a_{20} \Big|_{m_1=m_2=0} = 4 \left\{ (1-k)^2 (1+f(a)^2)^2 - 4f(a)^2 \left[\sin^2(a) (1-k)^2 + \cos^2(a) (1+k)^2 \right] \right\}.$$

The right hand side of (4.27) equals

$$4 \left[(1+f(a)^2)^2 - 4f(a)^2 \right] > 0 \quad \text{at} \quad k = 0,$$

$$-64f(a)^2 \cos^2(a) < 0 \quad \text{at} \quad k = 1,$$

and its derivative with respect to k is

$$(4.28) \quad \begin{aligned} \frac{\partial}{\partial k} a_{20} \Big|_{m_1=m_2=0} &= 8 \left\{ (k-1) (1+f(a)^2)^2 - 4f(a)^2 [\sin^2(a) (k-1) + \cos^2(a) (k+1)] \right\} \\ &= 8 \left\{ -(1-k) \left[(1+f(a)^2)^2 - 4f(a)^2 \sin^2(a) \right] - 4f(a)^2 \cos^2(a) (k+1) \right\} < 0, \end{aligned}$$

so that (ii) immediately follows.

Finally we prove (iii). At this point it is clear that all we need to show is that the zero that is at $\lambda = i \frac{\pi}{2a}$ when $k = k_0$ is entering $S_{\frac{\pi}{2a}}$ as k decreases. By lemma 4.6, this zero is of the form

$$(4.29) \quad \lambda = i \left(\frac{\pi}{2a} - x \right)$$

and, using (4.28) (with $k = k_0$), $x = x(k)$ satisfies

$$x(k_0) = 0 \quad , \quad x'(k_0) < 0 \quad ,$$

which, together with (4.29), gives the desired result. $\square$

LEMMA 4.8 (Case $m_1 = 0$, $m_2 \geq 0$; $0 < a < \frac{\pi}{2}$). Fix $0 < a < \frac{\pi}{2}$, $0 \leq k \leq 1$ and $m_1 = 0$. Write

$$(4.30) \quad a_{20} \Big|_{m_1=0} = -\alpha(m_2)^2 + \beta m_2 + \gamma$$

where $\alpha = \alpha(a)$, $\beta = \beta(a, k)$ and $\gamma = \gamma(a, k)$.

Let m_2^1, m_2^0 be the roots of

$$a_{20} \Big|_{m_1=0} = 0$$

($m_2^j = m_2^j(a, k)$, $j = 0, 1$), and assume $m_2^1, m_2^0 \in \mathbf{R}$, $m_2^0 \geq m_2^1$. Then

(i) if $k < k_0$ (k_0 is defined in lemma 4.7(ii)),

$$(4.31) \quad m_2^1 < 0.$$

(ii) There exists $\epsilon = \epsilon(a, k) > 0$ such that, if $|m_2 - m_2^0| < \epsilon$,

$$(4.32) \quad N\left(\frac{\pi}{2a}\right)(a, k, 0, m_2) = \begin{cases} N\left(\frac{\pi}{2a}\right)(a, k, 0, m_2^0) & \text{if } m_2 \geq m_2^0, \\ N\left(\frac{\pi}{2a}\right)(a, k, 0, m_2^0) + 1 & \text{if } m_2^1 < m_2 < m_2^0. \end{cases}$$

Proof. From (2.35) and (2.1)-(2.4) we find that

$$(4.33) \quad \begin{aligned} \alpha &= 16 (f(a)^2 - 1), \\ \beta &= 16 \{ -(1-k)(1+f(a)^2) + 2f(a)^2 [\sin^2(a)(1-k) + \cos^2(a)(1+k)] \}, \\ \gamma &= 4 \left\{ (1-k)^2 (1+f(a)^2)^2 - 4f(a)^2 [\sin^2(a)(1-k)^2 + \cos^2(a)(1+k)^2] \right\}. \end{aligned}$$

Clearly, from (2.37),

$$(4.34) \quad \alpha > 0.$$

Also, from (4.33),

$$\beta > 16(1-k)(f(a)^2 - 1)$$

and therefore

$$(4.35) \quad \beta > 0.$$

Finally, lemma 4.7(ii) implies

$$(4.36) \quad \begin{aligned} \gamma &< 0 & \text{if } k_0 < k, \\ \gamma &> 0 & \text{if } k_0 > k, \end{aligned}$$

and, since

$$m_2^1 = \frac{\beta - \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha},$$

(i) follows from (4.34)-(4.36).

Next, notice that, since $m_2^0 \geq m_2^1$, we have

$$(4.37) \quad \frac{\partial}{\partial m_2} d_2(0, a, k, 0, m_2^0) = \frac{\partial}{\partial m_2} a_{20} \Big|_{m_2=m_2^0} \leq 0$$

and

$$(4.38) \quad \frac{\partial^2}{\partial m_2^2} d_2(0, a, k, 0, m_2^0) = \frac{\partial^2}{\partial m_2^2} a_{20} \Big|_{m_2=m_2^0} = -2\alpha < 0.$$

But, by lemma 4.6, the zero which is at $\lambda = i\frac{\pi}{2a}$ when $m_2 = m_2^0$ is of the form

$$\lambda = i \left(\frac{\pi}{2a} - x \right),$$

where $x = x(m_2)$ satisfies, due to (4.37),(4.38),

$$\begin{aligned} x(m_2^0) &= 0, \quad x'(m_2^0) \leq 0 \quad \text{and} \\ x''(m_2^0) &< 0 \quad \text{if} \quad x'(m_2^0) = 0. \end{aligned}$$

This implies (ii). $\square$

LEMMA 4.9. Fix $0 < a < \frac{\pi}{2}$, $0 \leq k \leq 1$, $0 \leq m_2 \leq \frac{1}{2}$. Write

$$(4.39) \quad a_{20} = -\alpha(m_1)^2 + \varphi m_1 + \psi$$

where $\alpha = \alpha(a)$, $\varphi = \varphi(a, k, m_2)$ and $\psi = \psi(a, k, m_2)$.

Let m_1^1, m_1^0 be the roots of

$$a_{20} = 0$$

($m_1^j = m_1^j(a, k, m_2)$, $j = 0, 1$), and assume $m_1^1, m_1^0 \in \mathbf{R}$, $m_1^0 \geq m_1^1$. Then

$$(i) \text{ if } \psi = a_{20} \Big|_{m_1=0} > 0,$$

$$(4.40) \quad m_1^1 < 0.$$

(ii) There exists $\epsilon = \epsilon(a, k, m_2) > 0$ such that, if $|m_1 - m_1^0| < \epsilon$,

$$(4.41) \quad N\left(\frac{\pi}{2a}\right)(a, k, m_1, m_2) = \begin{cases} N\left(\frac{\pi}{2a}\right)(a, k, m_1^0, m_2) & \text{if } m_1 \geq m_1^0, \\ N\left(\frac{\pi}{2a}\right)(a, k, m_1^0, m_2) + 1 & \text{if } m_1^1 < m_1 < m_1^0. \end{cases}$$

Proof. The proof is similar to that of lemma 4.8. Again from (2.35) and (2.1)-(2.4),

$$\begin{aligned}
(4.42) \quad & \alpha = 16 (f(a)^2 - 1) , \\
& \varphi = 16k \{ (1-k) (1 + f(a)^2) - 2m_2 \\
& \quad - 2f(a)^2 [\sin^2(a) (1-k-m_2) + \cos^2(a) (m_2-k-1)] \} , \\
& \psi = a_{20} \Big|_{m_1=0} = 4 \left\{ [k-1+2m_2+(k-1)f(a)^2]^2 \right. \\
& \quad \left. - 4f(a)^2 [\sin^2(a) (k-1+m_2)^2 + \cos^2(a) (k+1-m_2)^2] \right\} .
\end{aligned}$$

Since $\alpha > 0$ and

$$m_1^1 = \frac{\varphi - \sqrt{\varphi^2 + 4\alpha\psi}}{2\alpha} ,$$

(4.40) is clear when $\psi > 0$.

Part (ii) is proved exactly as in the previous lemma $\square$

Proof of Theorem 4.2. Fix a , $0 < a < \frac{\pi}{2}$ and recall ((1.22) and lemma 2.1) that as the parameters k, m_1, m_2 vary, the zeros of G can cross $\partial Q_{\frac{\pi}{2a}, c}$ only through $\lambda = i\frac{\pi}{2a}$. In particular, if $m_1 = m_2 = 0$, lemma 4.7 implies

$$(4.43) \quad N\left(\frac{\pi}{2a}\right)(a, k, 0, 0) = \begin{cases} 0 & \text{if } k_0 \leq k \leq 1, \\ 1 & \text{if } 0 \leq k < k_0. \end{cases}$$

Next, for each k fixed and keeping $m_1 = 0$, we start increasing $m_2 \geq 0$. Then, (4.43) and lemma 4.8 imply:

(1) if $k < k_0$

$$N\left(\frac{\pi}{2a}\right)(a, k, 0, m_2) = \begin{cases} 1 & \text{if } 0 \leq m_2 < m_2^0 = m_2^0(a, k), \\ 0 & \text{if } m_2 \geq m_2^0 \end{cases}$$

and

(2) if $k \geq k_0$

$$N\left(\frac{\pi}{2a}\right)(a, k, 0, m_2) \leq 1$$

and

$$N\left(\frac{\pi}{2a}\right)(a, k, 0, m_2) = 0 \quad \text{if } m_2 \leq m_2^1 \text{ or } m_2^0 \leq m_2.$$

Combining (1) and (2), we conclude that

$$\begin{aligned}
(4.44) \quad & N\left(\frac{\pi}{2a}\right)(a, k, 0, m_2) \leq 1 \quad \text{and} \\
& N\left(\frac{\pi}{2a}\right)(a, k, 0, m_2) = 0 \quad \text{if } \psi \leq 0 ,
\end{aligned}$$

where $\psi = a_{20} \Big|_{m_1=0}$. (Note that $\psi \leq 0 \Leftrightarrow m_2 \leq m_2^1$ or $m_2^0 \leq m_2$).

Finally, for each (k, m_2) fixed we increase m_1 . Then, (4.3) follows from (4.44) and lemma 4.9. $\square$

§5. Numerical results. In the following graphs, the zeros of Δ were found using Newton's method. Figs. 5.1-5.7 correspond to the case $\frac{\pi}{2} < a < \pi$ while, in Figs. 5.8-5.10, $0 < a < \frac{\pi}{2}$. The imaginary part of the purely imaginary zeros $\lambda = iy$ is plotted against the angle a , for several choices of the parameters, in Figs. 5.1-5.5, 5.8 and 5.9. Thus, the critical points of a curve $a = a(y)$ in any of these graphs correspond to double zeros of Δ on the imaginary axis. Upon varying a , these zeros will leave the imaginary axis. This departure is described in Figs. 5.6 and 5.10 for the (double) zeros marked “+” in Figs. 5.5 and 5.9, respectively. We graph the zero $\lambda = x + iy$ in the complex plane as a curve given parametrically by $\lambda = \lambda(a)$ (for k, m_1, m_2 fixed). We only show the situation for $Re(\lambda) > 0$ since the location of the zeros is symmetric with respect to the imaginary axis. Fig. 5.7 provides another view of the behavior of the zeros in the example corresponding to Figs. 5.5, 5.6, when the angle a is near the angle for which a double zero occurs.

Finally, for clarity in the verification of the results in §§3,4,

$$\text{the curve marked “*” corresponds to } \begin{cases} x = \frac{\pi}{a} & \text{if } \frac{\pi}{2} < a < \pi, \\ x = \frac{\pi}{2a} & \text{if } 0 < a < \frac{\pi}{2}. \end{cases}$$

Fig. 5.1. $km_1 > m_2$. Two zeros in $S_{\frac{\pi}{2}}$ (Thm. 4.1).

Fig. 5.2. $m_2 - 2(1 - k) < km_1 < m_2$. Three zeros in $S_{\frac{\pi}{2}}$ if $\frac{\pi}{2} < a < a_0 = 2.0459$; two zeros in $S_{\frac{\pi}{2}}$ if $a_0 \leq a \leq \pi$ (Thm. 4.1).

Fig. 5.3. $km_1 > m_2$. Two zeros in $S_{\frac{\pi}{2}}$ (Thm. 4.1).

Fig. 5.4. $m_2 - 2(1 - k) < km_1 < m_2$. Three zeros in $S_{\frac{\pi}{2}}$ if $\frac{\pi}{2} < a < a_0 = 1.6284$; two zeros in $S_{\frac{\pi}{2}}$ if $a_0 \leq a \leq \pi$ (Thm. 4.1).

Fig. 5.5. $m_2 - 2(1 - k) > km_1$. Two zeros in $S_{\frac{\pi}{2}}$ (Thm. 4.1). A double zero occurs at $y = 0.5757$, $a = 2.7732$ (marked "+").

Fig. 5.6. Complex zero $\lambda = \lambda(a) = x(a) + iy(a)$ ($2.7732 < a < \pi$)
with $\lambda \Big|_{a=2.7732} = i0.5757$ (see Fig. 5.5).

Fig. 5.7. Another view of the case in Figs. 5.5, 5.6, for $a \approx 2.7732$.
Transition from two purely imaginary zeros to two zeros with opposite
real parts.

Fig. 5.8. One zero in $S_{\frac{\pi}{2a}}$ (Thm. 4.2).

Fig. 5.9. No zeros in $S_{\frac{\pi}{2a}}$ (Thm. 4.2). A double zero occurs at $y = 1.7693, a = 1.2724$ (marked "+").

Fig. 5.10. Complex zero $\lambda = \lambda(a) = x(a) + iy(a)$ ($1.2724 > a > 0.21$)

with $\lambda \Big|_{a=1.2724} = i1.7693$ (see Fig. 5.9).

Concluding remarks. Having rigorously established that Δ does not have more than three zeros λ (counted with multiplicities) with $0 < \text{Im}(\lambda) < 1$ it follows that no more than *three* terms in (1.24) will yield unbounded stresses as $r \rightarrow 0$. However, we should point out that in all the cases that were computed numerically the number of zeros of Δ in S_1 never exceeded *two*.

Regarding the number of higher order singularities in (1.24), the arguments leading to the results in Theorem 1.3 show that it depends not only on the characteristic exponents but also on the structure of the data f_i, g_i near $r = 0$. For example, if f_i and g_i vanish to sufficiently high order at $r = 0$, the last sum in (1.24) drops out (see Remark 1.4) and Theorem 4.2 implies that there is at most *one* term in the expansion provided $1 + m - \delta < \frac{\pi}{2a}$ (so that m can be large if a is small).

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