ERGODIC PROPERTIES OF LINEAR DYNAMICAL SYSTEMS

By

RUSSELL A. JOHNSON

KENNETH J. PALMER

AND

GEORGE R. SELL

IMA Preprint Series # 65

April 1984

INSTITUTE FOR MATHEMATICS AND ITS APPLICATIONS
UNIVERSITY OF MINNESOTA

514 Vincent Hall
206 Church Street SE.
Minneapolis, Minnesota 55455

Recent IMA Preprints

Preprint #	Author(s)	Title
1		Workshop Summaries from the September 1982 workshop on Statistical Mechanics, Dynamical Systems and Turbulence
2	Raphael De laLlave	A Simple Proof of C. Siegel's Center Theorem
3	H. Simpson, S. Spector	On Copositive Matrices and Strong Ellipticity for Isotropic Elastic Materials
4	George R. Sell	Vector Fields in the Vicinity of a Compact Invariant Manifold
5	Milan Miklavcic	Non-linear Stability of Asymptotic Suction
6	Hans Weinberger	A Simple System with a Continuum of Stable Inhomogeneous Steady States
7	Bau-Sen Du	Period 3 Bifurcation for the Logistic Mapping
8	Hans Weinberger	Optimal Numerical Approximation of a Linear Operator
9	L.R. Angel, D.F. Evans, B. Ninham	Three Component Ionic Microemulsions
10	D.F. Evans, D. Mitchell, S. Mukherjee, B. Ninham	Surfactant Diffusion; New Results and Interpretations
11	Leif Arkerlyd	A Remark about the Final Aperiodic Regime for Maps on the Interval
12	Luis Magalhaes	Manifolds of Global Solutions of Functional Differential Equations
13	Kenneth Meyer	Tori in Resonance
14	C. Eugene Wayne	Surface Models with Nonlocal Potentials: Unner Bounds
15	K.A. Pericak-Spector	On Stability and Uniqueness of Fluid Flow Through a Rigid Porous Medium
16	George R. Sell	Smooth Linearization Near a Fixed Point
17	David Wollkind	A Nonlinear Stability Analysis of a Model Equation for Alloy Solidification
18	Pierre Collet	Local C^∞ Conjugacy on the Julia Set for some Holomorphic Perturbations of $z + z^2$
19	Henry C. Simpson Scott J. Spector	On the Modified Bessel Functions of the First Kind On Barrelling for a Material in Finite Elasticity
20	George R. Sell	Linearization and Global Dynamics
21	P. Constantin C. Foias	Global Lyapunov Exponents, Kaplan-Yorke Formulas and the Dimension of the Attractors for 2D Navier-Stokes Equations
22	Milan Miklavcic	Stability for Semilinear Parabolic Equations with Noninvertible Linear Operator
23	P. Collet, H. Epstein G. Gallavotti	Perturbations of Geodesic Flows on Surfaces of Constant Negative Curvature and their Mixing Properties
24	J.E. Dunn, J. Serrin	On the Thermomechanics of Interstitial working
25	Scott J. Spector	On the Absence of Bifurcation for Elastic Bars in Uniaxial Tension
26	W.A. Coppel	Maps on an Interval
27	James Kirkwood	Phase Transitions in the Ising Model with Traverse Field
28	Luis Magalhaes	The Asymptotics of Solutions of Singularly Perturbed Functional Differential Equations: and Concentrated Delays are Different
29	Charles Tresser	Homoclinic Orbits for Flow in R^3
30	Charles Tresser	About some Theorems by L.P. Sil'nikov
31	Michael Aizermann	On the Renormalized Coupling Constant and the Susceptibility in ϕ_4^4 Field Theory and the Ising Model in Four Dimensions
32	C. Eugene Wayne	The KAM Theory of Systems with Short Range Interactions I
33	M. Slemrod J. E. Marsden	Temporal and Spatial Chaos in a Van der Waals Fluid due to Periodic Thermal Fluctuations

(continued on back cover)

Ergodic Properties of Linear Dynamical Systems

by

Russell A. Johnson*
Department of Mathematics
University of Southern California
Los Angeles, California 90089

Kenneth J. Palmer**
Department of Mathematics and Computer Science
University of Miami
Coral Gables, Florida 33124

George R. Sell
School of Mathematics
Institute for Mathematics and its Applications
University of Minnesota
Minneapolis, Minnesota 55455

This research was supported in part by a grant from the National Science Foundation

* Current Address: Sonderforschungsbereich 123
Universitat Heidelberg
6900 Heidelberg, FDR

** Current Address: Mathematics Department, University of Melbourne
Parkville, Victoria 3052, AUSTRALIA

Ergodic Properties of Linear Dynamical Systems

I. Introduction

Nearly two decades ago Oseledec (1968) published his proof of the Multiplicative Ergodic Theorem. This theorem, which is one of the milestones in the study of ergodic properties of dynamical systems, has had far-reaching applications, including its role in the work of Margulis (1975) on arithmeticity in Lie groups, in the theory of Pesin (1977) on Bernoullian substructures for diffeomorphisms, in the theory of Katok (1980) on entropy and periodic points, in the study of Kotani (1982) on spectral measures for Schroedinger operators, in the work of Constantin and Foias (1983) on attractors in the Navier-Stokes equations, and in the study of Novikov (1975) and Millionscikov (1978) on almost reducible systems with almost periodic coefficients. As a testimony to the importance of this theorem one finds several alternate proofs including the contemporaneous paper of Millionscikov (1968), and those of Raghunathan (1979), Ruelle (1979), and Crauel (1981), as well as the anticipatory paper of Liao (1966).

The Multiplicative Ergodic Theorem gives information about the dynamical structure of a cocycle ϕ , or a linear skew product flow π , over a suitable base space M . In typical applications the base space M is either an attractor; a compact invariant set; or the space of coefficients for a diffeomorphism, a differential equation, or a vector field. This theorem asserts that for every invariant probability measure μ on M there is a measurable decomposition of the vector bundle over M into invariant measurable subbundles, and that every solution with initial conditions in any of these subbundles has strong Lyapunov exponents. These exponents, or growth rates, depend on the measure μ , and when μ is ergodic, they are constant (almost everywhere) on M and form a finite set $\text{meas}\Sigma(\mu)$, the measurable (Millionscikov-Oseledec) spectrum.

The main objective in this paper is to study the connection between the measurable spectrum $\text{meas}\Sigma(\mu)$ and the dynamical spectrum $\text{dyn}\Sigma$ introduced by Sacker and Sell (1975, 1978, 1980). (Also see Daletskii and Krein (1974), as well as Selgrade (1975). The dynamical spectrum $\text{dyn}\Sigma$ consists of those values $\lambda \in \mathbb{R}$ for which the shifted flow π_λ fails to have an exponential dichotomy over M . It follows from the Spectral Theorem, Sacker and Sell (1978), that the dynamical spectrum is the finite union of disjoint compact intervals when M is compact and dynamically connected.

The dynamical spectrum and the theory of exponential dichotomies are central concepts in wide-ranging branches of analysis including the perturbation theories for invariant manifolds (see Sacker (1969), Fenichel (1971) and Hirsch, Pugh, and Shub (1977)), the bifurcation theories of Chenciner and Iooss (1979) and Sell (1979), the characterization of the spectrum of Schroedinger operators in Johnson (1982), linearization theories near invariant manifolds in Sell (1985), the study of transversal homoclinic orbits in Palmer (1984), as well as the study of inertial manifolds for dissipative systems in Foias, Sell and Temam (1985).

It is important therefore to understand the connection between these two spectral concepts. We will show, in Section VIII, that

$$(1.1) \quad \text{Boundary } \text{dyn}\Sigma \subseteq \bigcup_{\mu} \text{meas}\Sigma(\mu) \subseteq \text{dyn}\Sigma ,$$

where the union above is over all ergodic measures μ on M . We actually derive much more than (1.1). We show that the measurable invariant subbundles which arise in the Multiplicative Ergodic Theorem form a refinement of the continuous invariant subbundles described in the Spectral Theorem. The relationship (1.1) also leads to good methods for computing the Lyapunov exponents and the continuous spectral bundles, see Perry (1985).

Another objective is to show that the cocycle Φ , itself, has a strong Lyapunov exponent (almost everywhere) and that this agrees with $\max \text{meas}\Sigma(\mu)$. Although simple, this fact is very important because it forms the foundation for

deriving an approximation theory which leads to the numerical evaluation of the measurable and dynamical spectra. The approximation theory and the related numerical coding is described in the University of Minnesota Ph.D. thesis of David Perry (1985).

While doing this investigation we discovered a new proof of the Multiplicative Ergodic Theorem. Since our proof is a substantial simplification over other arguments we present it here. In addition to this simplification, our proof has some interesting geometrical features which may be useful elsewhere. While our proof of the Multiplicative Ergodic Theorem is restricted to cocycles over a compact base space M , we will see that this includes practically every application. Among other things, our theory applies to linear stochastic differential equations with bounded measurable coefficients, as well as to the linearized flow near an attractor in a nonlinear dynamical system.

A final objective of this paper is the presentation of several applications of these spectral theories. One of these theories, the theory of Lyapunov exponents for "spiral" systems, is central to any numerical investigation of Lyapunov exponents. Other applications include products of "random" matrices, stochastic differential equations, and the almost periodic Schroedinger operator.

This paper is organized as follows: In Section II we present the statements of the main theorems in this paper. Section III is concerned with a number of technical details which shall be used in the proofs of our theorems. One may wish to skip this on the first reading. In Section IV we present the basic triangularization method as it applies to linear skew product flows. Section V is concerned with a brief review of some basic facts about invariant measures, and in Section VI we present our proof of the Multiplicative Ergodic Theorem. The ergodic properties of the induced flow on the projective bundle are presented in Section VII. In Section VIII we derive (1.1) which describes

the connection between the measurable and the dynamical spectra. In Section IX we study the theory of wedge-product flows and show how this can be used to compute the measurable spectrum, and in Section X we present the applications discussed above. The paper concludes with an Appendix which contains some comments on related geometric properties of linear skew product flows.

II. Statement of Main Theorems

Let M be a compact Hausdorff space and let T denote either the integers $\mathbb{Z}$ or the reals $\mathbb{R}$. Assume that $\theta \cdot t$ is a flow on M , i.e. the mapping $(\theta, t) \mapsto \theta \cdot t$ of $M \times T$ into M is continuous and satisfies $\theta \cdot 0 = \theta$, and $\theta \cdot (t+s) = (\theta \cdot t) \cdot s$. The Krylov-Bogoliubov Theorem, see Nemytskii and Stepanov (1960), assures us that there is an invariant probability measure μ on M . This means that $\mu(A \cdot t) = \mu(A)$ for all Borel sets $A \subseteq M$ and all $t \in T$, where $A \cdot t = \{\theta \cdot t : \theta \in A\}$. The invariant measure μ is ergodic if $\mu(A \Delta A \cdot t) = 0$ for all $t \in T$ implies that $\mu(A) = 0$ or $\mu(A) = 1$. Recall that $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference. For an integer $m \geq 1$ let $\textcircled{\text{GL}}(m)$ denote the group of all isomorphisms on $\mathbb{R}^m$, i.e., the group of nonsingular $(m \times m)$ matrices with entries in $\mathbb{R}$. A cocycle on M is a continuous mapping $\phi: M \times T \rightarrow \textcircled{\text{GL}}(m)$ that satisfies

$$(2.1) \quad \phi(\theta, t+s) = \phi(\theta \cdot t, s) \phi(\theta, t)$$

for all $\theta \in M$ and $s, t \in T$. We note that ϕ is a cocycle on M if and only if

$$(2.2) \quad \pi(x, \theta, t) := (\phi(\theta, t)x, \theta \cdot t)$$

is a linear skew product flow on $\mathbb{R}^m \times M$.

If $T = \mathbb{R}$ we shall say that the flow π is smooth provided the mapping

$$A: \theta \mapsto \left. \frac{d}{dt} \phi(\theta, t) \right|_{t=0}$$

exists and is continuous. In this case the cocycle $\phi(\theta, t)$ is simply the fundamental matrix solution of

$$(2.3) \quad x' = A(\theta \cdot t)x, \quad (x \in \mathbb{R}^m)$$

that satisfies $\phi(\theta, 0) = I$. This is the prototypical example of a cocycle.

Let ϕ and ψ be two cocycles on M with range in $\textcircled{GL}(m)$. We shall say that ϕ and ψ are cohomologous if there is a continuous mapping $F: M \rightarrow \textcircled{GL}(m)$ that satisfies

$$(2.4) \quad \phi(\theta, t) = F(\theta \cdot t) \psi(\theta, t) F(\theta)^{-1}$$

for all $\theta \in M$, $t \in T$, where (-1) denotes the matrix inverse.

Let ϕ be a cocycle on M . Let $x \in \mathbb{R}^m$ and $\theta \in M$ and define the four Lyapunov exponents $\lambda_1^\pm(x, \theta)$, $\lambda_S^\pm(x, \theta)$ by

$$\lambda_S^\pm(x, \theta) = \limsup_{t \rightarrow \pm\infty} \frac{1}{t} \log |\phi(\theta, t)x|, \quad \lambda_1^\pm(x, \theta) = \liminf_{t \rightarrow \pm\infty} \frac{1}{t} \log |\phi(\theta, t)x|.$$

If it happens that the following two limits exist and are equal

$$(2.5) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \log |\phi(\theta, t)x| = \lim_{t \rightarrow -\infty} \frac{1}{t} \log |\phi(\theta, t)x|,$$

then we shall denote the common value as $\lambda(x, \theta)$. In the future when we write the symbol $\lambda(x, \theta)$ this should be interpreted as an assertion that both limits in (2.5) exist and $\lambda(x, \theta)$ is the common value. In this case one says that (x, θ) has a strong Lyapunov exponent.

Let ϕ and ψ be two cohomologous cocycles on M that satisfy (2.4), and let $x = F(\theta)y$. Then one has

$$\limsup_{t \rightarrow \pm\infty} \frac{1}{t} \log |\phi(\theta, t)x| = \limsup_{t \rightarrow \pm\infty} \frac{1}{t} \log |\psi(\theta, t)y|$$

and

$$\lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\phi(\theta, t)x| = \lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\psi(\theta, t)y|.$$

In other words, cohomologous cocycles have the same Lyapunov exponents.

For $0 \leq k \leq m$ let $\textcircled{G}(m, k)$ denote the Grassman manifold of k -planes in

$\mathbb{R}^m$, and let $\bigoplus(m) = \bigcup_{k=0}^m \bigoplus(m,k)$ denote the disjoint union of these compact manifolds. For $k \in \{1, \dots, m\}$ we shall let $N(k)$ denote those vectors $\vec{m} = (m_1, \dots, m_k)$ with $1 \leq m_i$ and $m_1 + \dots + m_k = m$.

The first two theorems are statements of the Multiplicative Ergodic Theorem.

2.1 Theorem Let M be a compact Hausdorff space with a flow $\theta \cdot t$ and let μ be an invariant probability measure on M . Let ϕ denote a cocycle on M . Then there exist:

- (i) an invariant set $M_\mu \subseteq M$ with $\mu(M_\mu) = 1$;
- (ii) a measurable decomposition $M_\mu = \bigcup M_\mu(p)$ where each $M_\mu(p)$ is invariant and the union is taken over all pairs $p = (k, \vec{m})$ where $1 \leq k \leq m$, and $\vec{m} \in N(k)$;
- (iii) measurable mappings $\lambda_1, \dots, \lambda_k: M_\mu(p) \rightarrow \mathbb{R}$, where

$$\lambda_1(\theta) < \lambda_2(\theta) < \dots < \lambda_k(\theta)$$

for $\theta \in M_\mu(p)$; and

- (iv) measurable mappings $\bigoplus_i: M_\mu(p) \rightarrow \bigoplus(m, m_i)$, $1 \leq i \leq k$, where $\vec{m} = (m_1, \dots, m_k)$,

such that for $\theta \in M_\mu(p)$ one has:

- (v) $\{\bigoplus_1(\theta), \dots, \bigoplus_k(\theta)\}$ is linearly independent;
- (vi) $\mathbb{R}^m = \bigoplus_1(\theta) + \dots + \bigoplus_k(\theta)$; and
- (vii) if $x \in \bigoplus_i(\theta)$, $x \neq 0$, then $\lambda(x, \theta) = \lambda_i(\theta)$ for $1 \leq i \leq k$.

If, in addition, μ is an ergodic measure, then precisely one $M_\mu(p)$ is nonempty, and the mappings $\lambda_i: M_\mu \rightarrow \mathbb{R}$ are constant, $1 \leq i \leq k$.

The results in the last theorem extend readily to linear skew product flows on arbitrary vector bundles, Sacker and Sell (1978).

2.2 Theorem Let $\mathbb{E}$ be a vector bundle over a compact Hausdorff space M and let π be a linear skew product flow on $\mathbb{E}$. Let μ be an invariant probability measure on M . Then the conclusions of Theorems 2.1 remain valid where $\mathbb{W}_i$, $1 \leq i \leq k$, now assume values in the appropriate Grassman bundles over M .

The measurable spectrum $\text{meas}\Sigma(\mu)$ is defined to be the collection $\{\lambda_1, \dots, \lambda_k\}$ when μ is ergodic. The numbers $m_1, \dots, m_k$ are the multiplicities of the spectral values $\lambda_1, \dots, \lambda_k$. When μ is not ergodic, then the spectrum $\text{meas}\Sigma(\mu, \theta) = \{\lambda_1(\theta), \dots, \lambda_k(\theta)\}$ and the multiplicities $\{m_1, \dots, m_k\}$ depend on $\theta \in M_\mu$ and $\theta \in M_\mu(\rho)$. For an ergodic measure μ , the measurable bundle associated with a spectral value λ_i , $1 \leq i \leq k$ is

$$\mathbb{W}_i = \{(x, \theta) : x \in \mathbb{W}_i(\theta), \theta \in M_\mu\}.$$

If μ is not ergodic, then the measurable bundles are defined similarly on each of the invariant sets $M_\mu(\rho)$.

The next theorem compares the measurable spectrum and the measurable bundles with the dynamical (or continuous) spectrum and associated continuous spectral subbundles arising in the theory of exponential dichotomies in linear skew product flows, see Sacker and Sell (1978, 1980).

Let $\pi(x, \theta, t)$ be a linear skew product flow on $\mathbb{R}^m \times M$, where M is a compact, connected space, and for $\lambda \in \mathbb{R}$ let

$$(2.6) \quad \pi_\lambda(x, \theta, t) := (\phi_\lambda(\theta, t)x, \theta \cdot t)$$

be the shifted flow, where $\phi_\lambda(\theta, t) := e^{-\lambda t} \phi(\theta, t)$. Recall that π_λ has an exponential dichotomy over M if there is a (continuous) projector $P(x, \theta) = (P(\theta)x, \theta)$ on $\mathbb{R}^m$ and constants $K \geq 1$, $\alpha > 0$ such that

$$|\phi_\lambda(\theta, t)P(\theta)\phi_\lambda^{-1}(\theta, s)| \leq Ke^{-\alpha(t-s)}, \quad s < t$$

$$|\Phi_\lambda(\theta, t)[I - P(\theta)]\Phi_\lambda^{-1}(\theta, s)| \leq Ke^{-\alpha(s-t)}, \quad t < s$$

for all $\theta \in M$ and $s, t \in T$. The set $\lambda \in \mathbb{R}$ for which π_λ fails to have an exponential dichotomy over M is defined to be $\text{dyn}\Sigma$, the dynamical spectrum. The Spectral Theorem assures us that $\text{dyn}\Sigma = \bigcup_{i=1}^k [a_i, b_i]$ is the union of k -nonoverlapping compact intervals, where $1 \leq k \leq m$. Also corresponding to each spectral interval $[a_i, b_i]$ there is an invariant spectral subbundle $\mathcal{V}_i$ of $\mathbb{R}^m \times M$ with $\dim \mathcal{V}_i(\theta) \geq 1$, where $\mathcal{V}_i(\theta) = \{x \in \mathbb{R}^m : (x, \theta) \in \mathcal{V}_i\}$, $1 \leq i \leq k$, the spaces $\{\mathcal{V}_1(\theta), \dots, \mathcal{V}_k(\theta)\}$ are linearly independent and $\mathbb{R}^m \times M = \mathcal{V}_1 + \dots + \mathcal{V}_k$ (as a Whitney sum). The boundary of $\text{dyn}\Sigma$ is the finite collection of end points $\{a_1, \dots, a_k, b_1, \dots, b_k\}$. The next theorem, which describes the connection between the measurable and dynamical spectra, is proved in Section VII.

2.3 Theorem. Let π be a linear skew product flow on $\mathbb{R}^m \times M$ where M is compact and connected, and let $\text{dyn}\Sigma$ denote the dynamical spectrum of π . Then one has

$$(2.7) \quad \text{Boundary } \text{dyn}\Sigma \subseteq \bigcup_{\mu} \text{meas}\Sigma(\mu) \subseteq \text{dyn}\Sigma$$

where the union is either over all invariant probability measures μ on M or over all ergodic measures on M . Let μ be a given invariant probability measure on M and let $\text{meas}\Sigma(\mu, \theta) = \{\lambda_1(\theta), \dots, \lambda_k(\theta)\}$ be the measurable spectrum for $\theta \in M_\mu(p)$. Then for each λ_j there is precisely one spectral interval $[a_j, b_j]$ with $\lambda_j(\theta) \in [a_j, b_j]$ for all $\theta \in M_\mu(p)$. Also the associated measurable bundle $\mathcal{W}_j(\theta)$ satisfies $\mathcal{W}_j(\theta) \subseteq \mathcal{V}_j(\theta)$ for all $\theta \in M_\mu(p)$. Finally one has $\mathcal{V}_j(\theta) = \sum \mathcal{W}_j(\theta)$ for all $\theta \in M_\mu(p)$ where the summation is over all j with $\lambda_j(\theta) \in [a_j, b_j]$.

If $T = Z$, then the last theorem is valid when M is compact and "dynamically connected", where the latter means that M cannot be written as the union of two disjoint nonempty closed invariant sets. Also, as in the spirit of Theorem 2.2, we note that Theorem 2.3 extends to linear skew product flows on general vector bundles.

Our next theorem is concerned directly with the problem of computing the measurable spectrum $\text{meas}\Sigma(\mu, \theta)$. The point is that one is able to do this without computing the basis elements $e_1, \dots, e_m$. The key idea here is the notion of a wedge product, cf. Matshushima (1972). For $1 \leq k \leq m$ let $\Lambda^k \mathbb{R}^m$ denote the vector space generated by all k -fold wedge products $x_1 \wedge \dots \wedge x_k$ where $x_i \in \mathbb{R}^m$, $1 \leq i \leq k$. Recall that the wedge product $x_1 \wedge \dots \wedge x_k$ is linear in each factor and antisymmetric, i.e. $x \wedge y = -y \wedge x$. If $L: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is linear then this induces a linear mapping $\Lambda^k L$ on $\Lambda^k \mathbb{R}^m$ by the formula

$$\Lambda^k L(x_1 \wedge \dots \wedge x_k) := Lx_1 \wedge \dots \wedge Lx_k.$$

Since one has $\Lambda^k(LM) = (\Lambda^k L)(\Lambda^k M)$, we see that if $\phi(\theta, t)$ is a cocycle on M then $\Lambda^k \phi(\theta, t)$ is also a cocycle, for $1 \leq k \leq m$.

In the statement of the next theorem reference will be made to the notation of Theorem 2.1. In particular for $\theta \in M_\mu(p)$ the growth rates

$$\lambda_1(\theta) < \lambda_2(\theta) < \dots < \lambda_k(\theta)$$

with multiplicities $m_1, \dots, m_k$ will be rewritten in the form

$$(2.8) \quad \gamma_1(\theta) < \gamma_2(\theta) < \dots < \gamma_m(\theta)$$

where $\lambda_i(\theta)$ is repeated m_i -times in (2.8), $1 \leq i \leq k$.

2.4 Theorem. Let M be a compact Hausdorff space with a flow $\theta \cdot t$ and let μ be an invariant probability measure on M . Let ϕ denote a cocycle on M and adopt the conclusions and notation of Theorem 2.1.

Let $\gamma_1, \dots, \gamma_m$ satisfy (2.8) for $\theta \in M_\mu(p)$. Then for all $\theta \in M_\mu(p)$ one has:

- (i) $\lim_{t \rightarrow +\infty} \frac{1}{t} \log |\phi(\theta, t)| = \gamma_m(\theta),$
- (ii) $\lim_{t \rightarrow +\infty} \frac{1}{t} \log |\Lambda^k \phi(\theta, t)| = \gamma_{m+1-k}(\theta) + \dots + \gamma_m(\theta),$ for $2 \leq k \leq m$,
- (iii) $\lim_{t \rightarrow -\infty} \frac{1}{t} \log |\phi(\theta, t)| = \gamma_1(\theta),$ and
- (iv) $\lim_{t \rightarrow -\infty} \frac{1}{t} \log |\Lambda^k \phi(\theta, t)| = \gamma_1(\theta) + \dots + \gamma_k(\theta),$ for $2 \leq k \leq m$.

The last theorem extends to linear skew product flows on a vector bundle $\textcircled{E}$ over a compact Hausdorff space M . In this case the wedge product of vectors $x_1, \dots, x_k \in \textcircled{E}(\theta)$ forms a new bundle $\Lambda^k \textcircled{E}$, $1 \leq k \leq m$ over M . Also the flow π on $\textcircled{E}$ induces a flow $\Lambda^k \pi$ on $\Lambda^k \textcircled{E}$. This extension is a direct consequence of the proof of the last theorem together with Lemma 3.4 below. We will omit the details.

Remark 2.1 For simplicity of exposition we have formulated these theorems for cocycles with values in $\textcircled{GL}(m, \mathbb{R})$. The theorems are valid for cocycles with values in $\textcircled{GL}(m, \mathbb{C})$, and the proofs we give below extend with only trivial modifications.

III. Some Technicalities

Before we turn our attention to the proofs of the main theorems, we need to dispense with some technical details which will enable us to simplify our arguments. We begin with a proof of the following facts:

1. One can assume, without loss of generality, that the base space M is a compact metric space instead of a compact Hausdorff space. (This fact simplifies substantially some of the measure theoretic considerations.)

2. If $T = R$ one can assume that the cocycle ϕ is the fundamental solution matrix of an ordinary differential equation on M with continuous coefficients. We call such a cocycle smooth.

3. A linear skew product flow on an arbitrary vector bundle (E) over a compact Hausdorff space M can be imbedded into a linear skew product flow on $R^m \times M$ for some $m > 1$.

The argument in each of these three cases is based on the same principle, viz. one can show that the given flow is cohomologous to the desired flow. The resulting cohomology preserves all the desired properties of our main theorems. In particular if ϕ_1 and ϕ_2 are two cohomologous cocycles on a compact Hausdorff space M that satisfy

$$F(\theta \cdot t)\phi_1(\theta, t) = \phi_2(\theta, t)F(\theta)$$

where $F : M \rightarrow GL(m)$ is continuous, then as noted above ϕ_1 and ϕ_2 have the same collection of Lyapunov exponents. Furthermore (x, θ) has a strong Lyapunov exponent for ϕ_1 if and only if $(F(\theta)x, \theta)$ has a strong Lyapunov exponent for ϕ_2 . Thus F preserves the measurable spectrum $\text{meas}\Sigma(\mu, \theta)$ and it maps the measurable bundles of ϕ_1 onto those of ϕ_2 . In addition F preserves the dynamical spectrum, and it sets up a one-to-one correspondence

between the continuous spectral subbundles.

The situation is, in fact, more general. Let M_1 and M_2 be two compact Hausdorff spaces and let $f : M_1 \rightarrow M_2$ be a flow epimorphism. Next let ϕ_1 be a cocycle on M_1 ($i = 1, 2$) and let $F : M_1 \rightarrow \textcircled{\text{GL}}(m)$ satisfy

$$F(\theta_1 \cdot t) \phi_1(\theta_1, t) = \phi_2(f(\theta_1), t) F(\theta_1) .$$

Then ϕ_1 and ϕ_2 have the same measurable and dynamical spectra and F sets up a one-to-one correspondence between the associated spectral bundles.

Our first step is to show that we can replace a compact Hausdorff base space M_1 with a compact metric space M_2 . We use an argument of Ellis (1969).

3.1 Lemma. Let ϕ_1 be a cocycle over a compact Hausdorff space M_1 with a flow $\theta_1 \cdot t$. Then there is (i) a compact metric space M_2 with a flow $\theta_2 \cdot t$, (ii) an flow epimorphism $f : M_1 \rightarrow M_2$ and (iii) a cocycle ϕ_2 over M_2 such that $\phi_1(\theta_1, t) = \phi_2(f(\theta_1), t)$.

Proof. Let $\{t_n\}$ be a countable dense subset of T . Let $\textcircled{A}$ be the closed subalgebra of $\textcircled{C}(M_1, \mathbb{R})$ generated by all functions of the form $\{\theta_1 \rightarrow \phi_{ij}(\theta_1, t_n)\}$ where ϕ_{ij} are the components of ϕ . Then $\textcircled{A}$ is a separable subalgebra of $\textcircled{C}(M_1, \mathbb{R})$. Since $\textcircled{A}$ is closed it contains all mappings $\{\theta_1 \rightarrow \phi_{ij}(\theta_1, \tau)\}$ where $\tau \in T$; in fact $\textcircled{A}$ is also the closed subalgebra generated by all such mappings for $\tau \in T$. Because of the cocycle identity (2.1) we see that $\textcircled{A}$ is invariant, in the sense that if $g \in \textcircled{A}$ then $g_\tau \in \textcircled{A}$, where $g_\tau(\theta_1) = g(\theta_1 \cdot \tau)$. The Stone Theorem, cf. Hewitt and Ross (1963, pp. 483-484), says that $\textcircled{A} = \textcircled{C}(M_2, \mathbb{R})$, where M_2 is the maximal ideal space of $\textcircled{A}$. Since $\textcircled{A}$ is separable, M_2 is a compact metric space. Recall that M_2 can be realized as the space of equivalence classes $[\theta_1]$ where $\theta_1 \sim \tilde{\theta}_1$ provided $\phi_{ij}(\theta_1, t_n) = \phi_{ij}(\tilde{\theta}_1, t_n)$ for all i, j and all t_n . Note that if $\theta_1 \sim \tilde{\theta}_1$ then

$\theta_1 \cdot \tau \sim \tilde{\theta}_1 \cdot \tau$ for all $\tau \in T$. Consequently a flow on M_2 is given by $[\theta_1] \cdot \tau = [\theta_1 \cdot \tau]$. Also the mapping $f(\theta_1) := [\theta_1]$ from M_1 to M_2 is an epimorphism because (A) is invariant. Finally we see that for each $t \in T$ the cocycle $\phi_1(\theta_1, t)$ depends only on the equivalence class $[\theta_1]$. So we conclude the proof by defining ϕ_2 by $\phi_2([\theta_1], t) := \phi_1(\theta_1, t)$. Q.E.D.

The following lemma appears in Ellis and Johnson (1982), but we include a proof for completeness of exposition.

3.2 Lemma. Let ϕ be a cocycle over a compact Hausdorff space M with $T = \mathbb{R}$. Then ϕ is cohomologous to a smooth cocycle ψ over M , i.e. $\psi(\theta, t)$ is a fundamental matrix solution to $x' = A(\theta \cdot t)x$ where A is given by (3.1) below.

Proof. Let $V \subseteq GL(m)$ be a compact convex neighborhood of the identity I and choose $r > 0$ so that $\phi(\theta, t) \in V$ for all $\theta \in M$ and $0 < t < r$. Define $F(\theta) := \frac{1}{r} \int_0^r \phi(\theta, s) ds$. Then $F(\theta)$ is invertible, and it is easily verified that the cocycle

$$\psi(\theta, t) := F(\theta \cdot t) \phi(\theta, t) F(\theta)^{-1} = \frac{1}{r} \int_t^{t+r} \phi(\theta, s) ds F(\theta)^{-1},$$

which is cohomologous to ϕ , is the fundamental matrix solution to $x' = A(\theta \cdot t)x$ where

$$(3.1) \quad A(\theta) := \frac{1}{r} [\phi(\theta, r) - I] F(\theta)^{-1}. \quad \text{Q.E.D.}$$

The next lemma will allow us to conclude that the Multiplicative Ergodic Theorem 2.2 is valid for linear skew product flows on a vector bundle (E) over a compact Hausdorff space M . The same lemma shows that Theorems 2.3 and 2.4 extend to vector bundles as well. Before stating this we need to derive the following general fact concerning smooth approximations to continuous mappings on a compact invariant set.

3.3 Lemma. Let M be a compact Hausdorff space with a flow $\theta \cdot t$ and let $f : M \rightarrow N$ be a continuous mapping where N is a smooth compact Riemannian manifold. Then for every $\delta > 0$ there is a continuous function $g : M \rightarrow N$ with the following properties:

- (i) $\sup\{\text{dist}(f(\theta), g(\theta)) : \theta \in M\} < \delta$.
- (ii) for every $\theta \in M$, the mapping $\theta \mapsto \frac{d}{dt} g(\theta \cdot t)|_{t=0}$ of M into the tangent bundle TN is a continuous mapping in θ .

Proof: Let $\delta > 0$ be given. The Tubular Neighborhood Theorem, see Guillemin and Pollack (1974), assures us that for a sufficiently large $m > 1$ there is a smooth imbedding $h : N \rightarrow \mathbb{R}^m$, an open set $W \supseteq h(N)$ and a smooth retract $R : W \rightarrow h(N)$. Now choose $\eta > 0$ so that if $\phi_1, \phi_2 \in h(N)$ and $|\phi_1 - \phi_2| < \eta$ then $\text{dist}(h^{-1}(\phi_1), h^{-1}(\phi_2)) < \delta$. Next choose $\tau > 0$ so that

$$V(\theta) := \text{Co}\{h(f(\theta \cdot t)) : 0 \leq t \leq \tau\} \subseteq W$$

for every $\theta \in M$, where Co refers to the closed convex hull, and $|h(f(\theta)) - R(y)| < \eta$ for every $\theta \in M$ and $y \in V(\theta)$. We now define $g : M \rightarrow N$ by

$$g := h^{-1} \circ R\left(\frac{1}{\tau} \int_0^\tau h \circ f(\theta \cdot s) ds\right).$$

Since $\frac{1}{\tau} \int_0^\tau h f(\theta \cdot s) ds \in V(\theta)$ we see that $|f(\theta) - g(\theta)| < \delta$ for all $\theta \in M$. Furthermore it is easy to conclude that g is C^1 along trajectories and the mapping $\theta \mapsto \frac{d}{dt} g(\theta \cdot t)|_{t=0}$ is continuous. Q.E.D.

3.4 Lemma. Let (E) be a finite dimensional vector bundle over a compact Hausdorff base space M and let $\pi(x, \theta, t) = (\phi(\theta, t)x, \theta \cdot t)$ be a linear skew product flow on (E) . Then for any $\lambda \in \mathbb{R}$ there exists an integer $m > 1$, a monomorphism $H : (E) \rightarrow \mathbb{R}^m \times M$, a smooth cocycle $\psi : M \rightarrow \text{GL}(m)$ and an orthogonal invariant resolution of the identity $Q = (Q_1, Q_2)$ such that $H(E) = \text{Range } Q_1$ and

$$\begin{aligned} Q_1(\theta \cdot t) \Psi(\theta, t) &= \Psi(\theta, t) Q_1(\theta), \quad H(\theta \cdot t) \Phi(\theta, t) = \Psi(\theta, t) H(\theta), \\ Q_2(\theta \cdot t) \Psi(\theta, t) &= \Psi(\theta, t) Q_2(\theta) = e^{\lambda t} Q_2(\theta). \end{aligned}$$

Proof. Since $\dim \mathbb{E}(\theta)$ is constant on the components of M , there is no loss in generality in assuming that $\dim \mathbb{E}(\theta) = k$ for all $\theta \in M$. The first step is to apply a standard result in the theory of vector bundles, Atiyah (1967, p. 25), which states that there is an integer $m > 0$ and an projector $P_1 : \mathbb{R}^m \times M \rightarrow \mathbb{R}^m \times M$ such that the vector bundle $\text{Range } P_1$ is isomorphic to $\mathbb{E}$. Let $\hat{H}: \mathbb{E} \rightarrow \text{Range } P_1 \subseteq \mathbb{R}^m \times M$ be the isomorphism. Without any loss of generality we can assume that $P_1(\theta)$ is an orthogonal projection on $\mathbb{R}^m$ for all $\theta \in M$. Let $P_2 := I - P_1$.

The mapping $\hat{W}_1: \theta \rightarrow \text{Range } P_1(\theta)$ defines a continuous mapping of M into the smooth manifold $\mathbb{G}(m, k)$ of k -planes in $\mathbb{R}^m$. By Lemma 3.3, there is a smooth mapping $W_1: M \rightarrow \mathbb{G}(m, k)$ that is close to $\hat{W}_1$. Define $Q_1(\theta)$ to be the orthogonal projection with $W_1(\theta) = \text{Range } Q_1(\theta)$. Since W_1 is smooth this means that $\theta \rightarrow \frac{d}{dt} Q_1(\theta \cdot t)|_{t=0}$ is continuous. Also $Q_2(\theta) = I - Q_1(\theta)$ is smooth. Since Q_1 is close to P_1 it follows that $H = Q_1 \hat{H}$ is an isomorphism of $\mathbb{E}$ onto $\text{Range } Q_1$.

Next we define a flow on $\mathbb{R}^m \times M$ under which Q_1 and Q_2 are invariant. Let $S(\theta) = Q_1'(\theta)Q_1(\theta) + Q_2'(\theta)Q_2(\theta)$ and let $\Psi_1(\theta, t)$ be the fundamental matrix solution of $x' = S(\theta \cdot t)x$ satisfying $\Psi_1(\theta, 0) = I$. Then as shown by Daletskii and Krein (1974) one has

$$Q_i(\theta \cdot t) \Psi_1(\theta, t) = \Psi_1(\theta, t) Q_i(\theta)$$

for all $\theta \in M$ and $t \in \mathbb{R}$. Define a cocycle Ψ on M by

$$\Psi(\theta, t)Q_1(\theta) = H(\theta \cdot t)\Phi(\theta, t)H^{-1}(\theta)$$

$$\Psi(\theta, t)Q_2(\theta) = e^{\lambda t} \Psi_1(\theta, t)Q_2(\theta) .$$

It is now straight forward to check the remaining details. Lemma 3.2 assures us that Ψ can be chosen to be smooth. Q.E.D.

IV. Triangularization of Cocycles

We turn next to the theory of the Gram-Schmidt factorization of isomorphisms on R^m , where R^m has the Euclidean inner product $\langle \cdot, \cdot \rangle$. Let $GL(m)$ denote the group of all isomorphisms of R^m . Each element $L \in GL(m)$ is identified with the $(m \times m)$ matrix whose column vectors satisfy $\text{col}_i L = Le_i$, $1 \leq i \leq m$, where $\{e_1, \dots, e_m\}$ is a fixed orthonormal basis in R^m . Let $O(m) = O(m)$ denote the subgroup of $GL(m)$ consisting of all orthogonal linear transformations, and let $T^+(m)$ denote the subcollection of all upper triangular matrices $L \in GL(m)$ with positive entries on the main diagonal. Then $T^+(m)$ is also a subgroup of $GL(m)$ and one has

$$(4.1) \quad O(m) \cap T^+(m) = \{I\}.$$

The Gram-Schmidt orthogonalization process assures us that for every $A \in GL(m)$ there are unique matrices $G(A) \in O(m)$ and $T(A) \in T^+(m)$ such that

$$(4.2) \quad G(A) = AT(A).$$

Since the entries in $T(A)$ are algebraic functions of $\langle \text{col}_i A, \text{col}_j A \rangle$ we see that both $T(A)$ and $G(A)$ are smooth functions of A .

Next we note that one has

$$(4.3) \quad G(AB) = G(AG(B)), \quad T(AB) = T(B)T(ABT(B)).$$

In order to prove (4.3) we define $U, V \in O(m)$ by

$$\begin{aligned} U &:= G(AB) = ABT(AB) \\ V &:= G(AG(B)) = ABT(B)T(ABT(B)), \end{aligned}$$

where (4.2) is used above. One then has

$$(4.4) \quad U^{-1}V = T(AB)^{-1}T(B)T(ABT(B)) \in O(m) \cap T^+(m).$$

Hence by (4.1) $U^{-1}V = I$, which proves (4.3).

Let $\phi: M \rightarrow \textcircled{GL}(m)$ be a cocycle on M . Then (4.2) admits the factorization

$$(4.5) \quad G(\phi(\theta, t)U) = \phi(\theta, t)U T(\phi(\theta, t)U)$$

for every $U \in \textcircled{O}(m)$. This permits us to define a new flow on $H := M \times \textcircled{O}(m)$

as follows: Let $\phi := (\theta, U) \in M \times \textcircled{O}(m)$ and define

$$(4.6) \quad \phi \cdot t := (\theta \cdot t, G(\phi(\theta, t)U)) .$$

4.1 Lemma. Equation (4.6) defines a flow on $H = M \times \textcircled{O}(m)$.

Proof. It suffices to verify the group property

$$G[\phi(\theta \cdot t, s)G(\phi(\theta, t)U)] = G(\phi(\theta, t+s)U) .$$

However, this is an immediate consequence of (2.1) and (4.3).

Q.E.D.

We noted in Section II that the cocycle $\phi(\theta, t)$ on M defines a linear skew product flow on $R^m \times M$ by $\pi(x, \theta, t) = (\phi(\theta, t)x, \theta \cdot t)$. By using (4.6) we see that π can be lifted to a new flow $\hat{\pi}$ on $R^m \times H$ by

$$\hat{\pi}(x, \phi, t) := (\phi(\theta, t)x, \phi \cdot t)$$

where $\phi = (\theta, U)$. Let $q: M \times \textcircled{GL}(m) \rightarrow \textcircled{GL}(m)$ and $r: M \times \textcircled{GL}(m) \rightarrow M$,

(or $r: H \rightarrow M$) be the natural projections. Define $\Psi(\phi, t)$ by

$$(4.7) \quad \Psi(\phi, t) := q(\phi \cdot t)^{-1} \phi(\theta, t) q(\phi)$$

where the (-1) denotes the matrix inverse and $\phi = (\theta, U)$. Since $\phi \cdot t$ is a flow on H , it follows that Ψ is a cocycle on H , and

$$(4.8) \quad \tilde{\pi}(x, \phi, t) := (\Psi(\phi, t)x, \phi \cdot t)$$

is a linear skew product flow on $R^m \times H$ which is cohomologous to $\hat{\pi}$. The

following lemma is now an immediate consequence of (4.5-7).

4.2 Lemma. Let ϕ be a cocycle on M and define $\phi \cdot t$ and $\Psi(\phi, t)$ by (4.6-7). Then one has

$$(4.9) \quad \Psi(\phi, t) = T(\phi(\theta, t)U)^{-1} \in \mathcal{O}^+(m)$$

for all $\phi = (\theta, U) \in H$ and $t \in T$.

Remarks. **4.3** The triangularization method described above is directly related to the familiar technique developed by Lyapunov (1892), Perron (1930) and Diliberto (1950). Let $T = \mathbb{R}$ and let $\phi(\theta, t)$ be a smooth cocycle and (therefore) the fundamental solution matrix of a differential equation

$$(4.10) \quad x' = A(\theta \cdot t)x, \quad x \in \mathbb{R}^m, \quad \theta \in M$$

where A is a continuous $(m \times m)$ matrix valued function defined on M . Then $\Psi(\phi, t)$ is the fundamental solution matrix of

$$(4.11) \quad y' = B(\phi \cdot t)y, \quad y \in \mathbb{R}^m, \quad \phi \in H$$

where $\phi = (\theta, U)$, $B = G^{-1}(AG - G')$, $G = G(\phi(\theta, t)U)$ and $G' = \frac{d}{dt} G$. The change of variables which maps solutions of (4.11) onto those of (4.10) is

$$x = P(t)y = G(\phi(\theta, t)U)y.$$

Also since the fundamental matrix solution of (4.11) is Ψ , an upper triangular matrix, we see that B is also upper triangular.

4.4 For $T = \mathbb{Z}$ this is basically the triangularization method described in Oseledec (1968).

V. Invariant Measures

In this section we record for reference a number of known results concerning invariant measures associated with the flows on M and H . Let r be the natural projection of H onto M . By (4.6) we see that r is a flow epimorphism, i.e. $r(\phi) \cdot t = r(\phi \cdot t)$.

Because of Lemma 3.1 we see that there is no loss in generality in assuming M (and therefore H) to be compact metric spaces. The Riesz Representation Theorem says that for any compact metric space M there is an isomorphism between bounded positive linear functionals λ on $\mathcal{C}(M, \mathbb{R})$ satisfying $\lambda(1) = 1$ with the (regular, positive, Borel, probability) measures μ on M , and this isomorphism is given by the formula

$$\lambda(f) = \int_M f(\theta) \mu(d\theta) .$$

Hereafter we will interchange freely such functionals and the associated measures and write $\mu(f)$ in place of $\lambda(f)$.

The measure μ is invariant for the flow $\theta \cdot t$ if and only if $\lambda(f_\tau) = \lambda(f)$ for all $f \in \mathcal{C}(M, \mathbb{R})$ and all $\tau \in T$ where $f_\tau(\theta) = f(\theta \cdot \tau)$. Also μ is ergodic if and only if for $f \in \mathcal{L}^1(M, \mathbb{R})$ one has

$$f_\tau = f \text{ for all } \tau \in T \iff f \equiv \text{constant} .$$

The Krylov-Bogoliubov method, cf. Nemytskii and Stepanov (1960), is a method for constructing invariant measures. Let us review this for the case $T = \mathbb{R}$. Let μ be a given measure on M and define

$$(5.1) \quad \mu_T(f) := \frac{1}{T} \int_0^T \mu(f_\tau) d\tau$$

for $T > 0$. Let $T_n \rightarrow +\infty$, and suppose (by choosing a subsequence if necessary) that μ_{T_n} converges weakly to a measure $\hat{\mu}$. Then $\hat{\mu}$ is easily seen to be invariant.

If the original measure μ is a δ -measure, i.e. $\mu(f) = \delta_\theta(f) = f(\theta)$, then (5.1) becomes

$$(5.2) \quad \mu_T(f) := \frac{1}{T} \int_0^T f(\theta \cdot \tau) d\tau$$

Notice that if the original measure μ has support in a closed invariant set M_0 , then the induced invariant measure μ has support in M_0 as well.

Let μ be a given invariant measure on M . Let $I(\mu)$ denote the collection of all invariant measures ν on H that cover μ , i.e. $\nu \in I(\mu)$ if it is invariant and $r(\nu) = \mu$. If μ is an ergodic measure on M we let $E(\mu)$ denote the ergodic measures $\nu \in I(\mu)$. By using the Krylov-Bogoliubov method we see that $I(\mu)$ is nonempty. Indeed if ℓ is any measure on $\mathcal{Q}(M)$, then $\mu \times \ell$ is a measure on H . Now form

$$(\mu \times \ell)_T(g) = \frac{1}{T} \int_0^T (\mu \times \ell)(g_\tau) d\tau,$$

and let ν be a resulting invariant measure. In order to show that ν covers μ we need to show that $\nu(f) = \mu(f)$ whenever $f = f(\theta)$ depends only on the coordinate $\theta \in M$. However in this case one has

$$(\mu \times \ell)(f_\tau) = \mu(f_\tau) = \mu(f) = (\mu \times \ell)_T(f)$$

since μ is invariant. Hence the limit ν satisfies $\nu(f) = \mu(f)$. Since $I(\mu)$ is nonempty, compact and convex it has extreme points. The extreme points in $I(\mu)$ are ergodic measures ν when μ is ergodic.

VI. Proof of the Multiplicative Ergodic Theorem

Throughout this section we will adopt without any loss of generality the following Standing Hypotheses which will lead to a proof of Theorems 2.1 and 2.2: Let $\pi(x, \theta, t) = (\phi(\theta, t)x, \theta \cdot t)$ be a given linear skew-product flow on the trivial (Lemma 3.4) vector bundle $\mathbb{R}^m \times M$, where M is a compact metric space (Lemma 3.1). If $T = \mathbb{R}$ we assume that $\phi(\theta, t)$ is smooth and is the fundamental solution matrix (Lemma 3.2) of

$$(6.1) \quad x' = A(\theta \cdot t)x \quad x \in \mathbb{R}^m, \theta \in M.$$

Let $\hat{\pi}(x, \phi, t) = (\Psi(\phi, t)x, \phi \cdot t)$ be the cohomologous triangular flow induced on $\mathbb{R}^m \times H$ (Section IV). If $T = \mathbb{R}$ then $\Psi(\phi, t)$ is also smooth and is the fundamental solution matrix of

$$(6.2) \quad y' = B(\phi \cdot t)y \quad y \in \mathbb{R}^m, \phi \in H.$$

where B is a continuous upper triangular matrix. Let μ be a given invariant measure on M and let $\nu \in I(\mu)$ be any invariant measure on H that covers μ . If μ is ergodic we assume that ν is ergodic (Section V). Also $r : H \rightarrow M$ is the natural projection.

We shall say that a point $\theta \in M$ (or $\phi \in H$) is a Lyapunov point for π (or $\hat{\pi}$) if there are real numbers $\gamma_1, \dots, \gamma_m$ and a basis $e_1, \dots, e_m$ of $\mathbb{R}^m$ such that

$$(6.3) \quad \lambda(e_i, \theta) := \lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\phi(\theta, t)e_i| = \gamma_i$$

$$(6.4) \quad (\text{or } \lambda(e_i, \phi) := \lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\Psi(\phi, t)e_i| = \gamma_i)$$

for $1 \leq i \leq m$.

Roughly speaking, the Multiplicative Ergodic Theorem asserts that there are many Lyapunov points, (i.e. $\mu(M_\mu) = 1$) and that they fit together in a measurable manner. As we now show this follows from the triangularization technique described in Section IV.

6.1 Lemma. Let $\phi = (\theta, U) \in H$ be a Lyapunov point for $\hat{T}$. Then $\theta \in M$ is a Lyapunov point for π .

Proof. Choose $\gamma_1, \dots, \gamma_m$ in $\mathbb{R}$ and a basis $e_1, \dots, e_m$ in $\mathbb{R}^m$ so that (6.4) is satisfied. Define $f_1, \dots, f_m$ by $f_i = Ue_i$, $1 \leq i \leq m$. Equations (4.5-6) and (4.9) give

$$(6.5) \quad G(\phi(\theta, t)U)\Psi(\phi, t) = \phi(\theta, t)U.$$

Since $G(\phi(\theta, t)U)$ is an orthogonal matrix one has

$$|\phi(\theta, t)f_i| = |\Psi(\phi, t)e_i|, \quad 1 \leq i \leq m.$$

It follows that (6.3) is satisfied with the same γ_i when e_i is replaced by f_i , $1 \leq i \leq m$. Q.E.D.

The next lemma is the key step in our proof.

6.2 Lemma. Let $\phi = (\theta, U) \in H$ be fixed. Assume that the diagonal entries $\Psi(\phi, t)$ satisfy

$$(6.6) \quad \lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\psi_{ii}(\phi, t)| = \gamma_i$$

for some constants γ_i , $1 \leq i \leq m$. Then ϕ is a Lyapunov point for $\hat{\pi}$ where the growth rates $\gamma_1, \dots, \gamma_m$ are given by (6.6), and the associated matrix V of basis vectors $\{e_1, \dots, e_m\}$ is an upper triangular matrix with $v_{ii} = 1$, $1 \leq i \leq m$.

If $T = \mathbb{R}$, then $\psi_{ij}(\phi, t) = \exp(\int_0^t b_{ij}(\phi \cdot s) ds)$ where b_{ij} , $1 \leq i \leq m$, are the diagonal entries of the triangular matrix B in (6.2). In this case (6.6)

becomes

$$\lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\psi_{ij}(\phi, t)| = \lim_{|t| \rightarrow \infty} \frac{1}{t} \int_0^t b_{ij}(\phi \cdot s) ds = \gamma_j, \quad 1 \leq i \leq m.$$

Also if $T = Z$, then the diagonal elements of ψ satisfy

$$\psi_{ij}(\phi, t) = \prod_{s=0}^{t-1} \psi_{ij}(\phi \cdot s, 1) \quad t > 0$$

with a similar expression valid for $t < 0$. For $t > 0$ one has

$$\frac{1}{t} \log |\psi_{ij}(\phi, t)| = \frac{1}{t} \sum_{s=0}^{t-1} \log |\psi_{ij}(\phi \cdot s, 1)|.$$

We see then that for both $T = R$ and $T = Z$, the limits in (6.6) are time-averages of continuous real-valued functions defined on H . This fact will be used later when we apply the Birkhoff Ergodic Theorem.

Proof. The argument we now give applies to any triangular cocycle ψ over any compact metric space H . We will not use the special form of the flow on H .

Let i satisfy $1 \leq i \leq m$. For any upper triangular $(m \times m)$ matrix T we let T_i denote the lower-right $(k \times k)$ -dimensional block where $k = (m - i + 1)$. Thus $T_1 = T$ and $T_m = (t_{mm})$. For the matrix B given by (6.2) we let β_i denote the $(m - i)$ -dimensional row vector that satisfies

$$B_i = \begin{pmatrix} b_{ii} & \beta_i \\ 0 & B_{i+1} \end{pmatrix}$$

for $1 \leq i \leq m-1$.

The upper triangular matrix V of basis vectors is obtained by constructing the V_i inductively starting with $V_m = (1)$. Suppose $1 \leq i \leq m-1$ and that V_{i+1} has been constructed with the properties that its diagonal elements are 1 and

$$\lambda(\text{col}_j(V_{i+1}), \phi) = \gamma_j$$

for $i+1 < j < m$. To construct V_i with the corresponding properties we first define $v_{ii} = 1$. For $1 < i < m-1$ and $i+1 < j < m$ we define*

$$(6.7) \quad v_{ij} = v_{ij}(\phi) := \int_{\tau}^0 \psi_{ii}^{-1}(\phi, s) \beta_i(\phi \cdot s) \psi_{i+1}(\phi, s) \text{col}_j(V_{i+1}(\phi)) ds$$

$$\text{col}_j(V_i) := \begin{pmatrix} v_{ij} \\ \text{col}_j(V_{i+1}) \end{pmatrix}$$

where

$$\tau = \tau_{ij} := \begin{cases} \infty, & \text{if } \gamma_i > \gamma_j \\ 0, & \text{if } \gamma_i = \gamma_j \\ -\infty, & \text{if } \gamma_i < \gamma_j \end{cases}.$$

Equations (6.4) and (6.6) and the induction hypothesis imply that for every $\varepsilon > 0$ there are constants K_1 and K_2 such that for $t > 0$ one has

$$\begin{aligned} |\psi_{ii}(\theta, t)| &\leq K_2 \exp[(\gamma_i + \varepsilon)t] \\ |\psi_{ii}^{-1}(\theta, t)| &\leq K_2 \exp[-(\gamma_i + \varepsilon)t] \\ K_1 \exp[(\gamma_j - \varepsilon)t] &\leq |\psi_{i+1}(\phi, t) \text{col}_j(V_{i+1})| \leq K_2 \exp[(\gamma_j + \varepsilon)t] \end{aligned}$$

for $i+1 < j < m$. Since $|\beta_i|$ is uniformly bounded on H , it follows that the infinite integral in (6.7) is well-defined.

The variation of constants formula for the block-triangular system

$u' = B_i(\phi \cdot t)u$ yields

$$(6.8) \quad \psi_i(\phi, t) \text{col}_j(V_i) = \begin{pmatrix} \psi_{ii}(\phi, t) \int_{\tau}^t \psi_{ii}^{-1}(\phi, s) \beta_i(\phi \cdot s) \psi_{i+1}(\phi, s) \text{col}_j(V_{i+1}) ds \\ \psi_{i+1}(\phi, t) \text{col}_j(V_{i+1}) \end{pmatrix}$$

*

We assume for the moment that $T = \mathbb{R}$. The modification of our argument needed for the case $T = \mathbb{Z}$ is described in the last paragraph of the proof.

for $i + 1 < j < m$ and $\Psi_i(\phi, t) \text{col}_i(V_i) = \text{col}(\psi_{ii}(\phi, t), 0, \dots, 0)$. Let $v_{ij}(t)$ denote the first entry in (6.8). One then has $\lambda(\text{col}_i(V_i), \phi) = \gamma_i$. Also it is known that

$$(6.9) \quad \lambda(\text{col}_j(V_i), \phi) = \lambda(\text{col}_j(V_{i+1}), \phi) = \gamma_j$$

for $i + 1 < j < m$, cf. Millionschikov (1968), but we shall include a proof for completeness. Indeed it follows from (6.8) and the inequalities after (6.7) that there is a constant K such that

$$|v_{ij}(t)| \leq K \exp[(\gamma_j + 3\varepsilon)t]$$

for $i + 1 < j < m$ and $t > 0$. Since ε is arbitrary one has

$$(6.10) \quad \limsup_{|t| \rightarrow +\infty} \frac{1}{|t|} \log |v_{ij}(t)| \leq \gamma_j, \quad i + 1 < j < m,$$

and therefore by (6.8) we have

$$\begin{aligned} \gamma_j &= \lim_{t \rightarrow +\infty} \frac{1}{t} \log |\Psi_{i+1}(\phi, t) \text{col}_j(V_{i+1})| \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log |\Psi_i(\phi, t) \text{col}_j(V_i)| \\ &\leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |\Psi_i(\phi, t) \text{col}_j(V_i)| \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |v_{ij}(t)| \leq \gamma_j. \end{aligned}$$

A similar argument applies as $t \rightarrow -\infty$. Also (6.10) is valid as $t \rightarrow -\infty$.

This completes the argument for $T = \mathbb{R}$. If $T = \mathbb{Z}$ the integrals in (6.7-8) are replaced by sums. For example by the variation of constants formula in Sacker and Sell (1976b), $v_{ij}(t)$ takes the form

$$v_{ij}(t) = \psi_{ii}(\phi, t) \sum_{s=\tau}^{t-1} \psi_{ii}^{-1}(\phi, s+1) \beta_i(\phi, s) \Psi_{i+1}(\phi, s) \text{col}_j(V_{i+1})$$

where $\beta_i = \beta_i(\phi)$ is the $(m-i)$ -dimensional row vector that satisfies

$$\Psi_i(\phi, 1) = \begin{pmatrix} \psi_{ii}(\phi, 1) & \beta_i(\phi) \\ 0 & \Psi_{i+1}(\phi, 1) \end{pmatrix}$$

for all $\phi \in H$. We will omit the details, which are easily verified. Q.E.D.

6.3 Lemma. Let $\phi = (\theta, U)$ satisfy the hypotheses of Lemma 6.2 and let V be the matrix of basis vectors constructed above. Then there are upper triangular $(m \times m)$ matrices $S(\phi, t)$ and $D(\phi, t)$ that satisfy

- (i) $\psi(\phi, t)V = S(\phi, t)D(\phi, t)$
- (ii) $D(\phi, t) = \text{diag}(\psi_{11}(\phi, t), \dots, \psi_{mm}(\phi, t))$
- (iii) $\limsup_{|t| \rightarrow \infty} \frac{1}{|t|} \log |S(\phi, t)| < 0$
- (iv) $\limsup_{|t| \rightarrow \infty} \frac{1}{|t|} \log |S^{-1}(\phi, t)| < 0$.

Proof. S and D are uniquely determined by (i) and (ii) and

$$S(\phi, t) = \begin{pmatrix} 1 & \psi_{22}^{-1}(\phi, t)v_{12}(t) & \dots & \psi_{mm}^{-1}(\phi, t)v_{1m}(t) \\ 0 & 1 & \dots & \psi_{mm}^{-1}(\phi, t)v_{2m}(t) \\ & & \dots & \\ 0 & 0 & & 1 \end{pmatrix}.$$

It follows from (6.10) that

$$\limsup_{|t| \rightarrow +\infty} \frac{1}{|t|} \log |\psi_{jj}^{-1}(\phi, t)v_{ij}(t)| < 0$$

for $1 \leq i \leq m-1$ and $i+1 \leq j \leq m$. Parts (iii) and (iv) then follow from the last inequality and the fact that the entries in S^{-1} are polynomials in the entries of S . Q.E.D.

Next we apply the Birkhoff Ergodic Theorem which assures us that there are a good supply of Lyapunov points in H .

6.4 Lemma. There is a Borel measurable invariant set $H_\nu \subseteq H$ with
 $\nu(H_\nu) = 1$ and such that every point $\phi \in H_\nu$ is a Lyapunov point.

Proof. As noted above the limits

$$\lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\psi_{ij}(\phi, t)|, \quad 1 \leq i \leq m$$

are time-averages of continuous functions defined on H . The Birkhoff Ergodic Theorem, see Nemytskii and Stepanov (1960), asserts that there is a Borel set $H_\nu \subseteq H$ with $\nu(H_\nu) = 1$, and there exist bounded Borel measurable invariant functions $\rho_1, \dots, \rho_m: H_\nu \rightarrow \mathbb{R}$ with the property that

$$(6.11) \quad \lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\psi_{ij}(\phi, t)| = \rho_i(\phi), \quad 1 \leq i \leq m$$

for all $\phi \in H_\nu$. It then follows from Lemma 6.2 that each $\phi \in H_\nu$ is a Lyapunov point. Q.E.D.

Let $\rho_1, \dots, \rho_m$ satisfy (6.11). For each integer k , $1 \leq k \leq m$, let $N(k)$ denote the collection of vectors $\vec{m} = (m_1, \dots, m_k)$ with integer entries that satisfy $1 \leq m_j$, $1 \leq j \leq k$, and $m_1 + \dots + m_k = m$. We will now construct a measurable decomposition of H_ν . Fix $\phi \in H_\nu$. We then note that there is an integer k , $1 \leq k \leq m$, and an $\vec{m} \in N(k)$ such that the following two properties hold:

(i) There are exactly k distinct values in the collection

$\{\rho_1(\phi), \dots, \rho_m(\phi)\}$, which we rewrite as $\{\lambda_1(\phi), \dots, \lambda_k(\phi)\}$ where

$$(6.12) \quad \lambda_1(\phi) < \lambda_2(\phi) < \dots < \lambda_k(\phi).$$

(ii) The cardinality of the set $\{i: 1 \leq i \leq m \text{ and } \rho_i(\phi) = \lambda_j(\phi)\}$ is m_j for each j , $1 \leq j \leq k$.

We denote the ordered pair $(k, \vec{m})$ briefly by p and define $H_v(p)$ to be the set of all $\phi \in H_v$ to which k and $\vec{m}$ correspond as above. The set $H_v(p)$ is Borel measurable since it is the pullback of the closed set

$$\{(x_1, \dots, x_m): x_1 = x_2 = \dots = x_{m_1} < x_{m_1+1} = \dots = x_{m_2} < \dots < x_{m_{k-1}+1} = \dots = x_{m_k}\}$$

by the Borel measurable function that is the composition of the Borel measurable function $\phi \mapsto (\rho_1(\phi), \dots, \rho_m(\phi))$ with the continuous function σ that maps $(x_1, x_2, \dots, x_m)$ onto its permutation $(x'_1, x'_2, \dots, x'_m)$ where $x'_1 < x'_2 < \dots < x'_m$.

It is easy to see that $H_v = \bigcup H_v(p)$, where the union is taken over all such points $p = (k, \vec{m})$, is a measurable decomposition of H_v . Since each ρ_i is invariant we see that the sets $H_v(p)$ are invariant. If v is an ergodic measure, then all but one $H_v(p)$ has v -measure 0. The following result is a consequence of the measurability and invariance of $\rho_1, \dots, \rho_m$.

6.5 Lemma. The functions $\lambda_1, \dots, \lambda_k: H_v(p) \rightarrow \mathbb{R}$ are Borel measurable and invariant.

Let $\phi \in H_v(p)$ and let $\rho_1, \dots, \rho_m$ and $\lambda_1, \dots, \lambda_k$ be given as above. Let $e_1, \dots, e_m$ be the basis in $\mathbb{R}^m$ constructed in Lemma 6.2. Thus one has $\lambda(e_i, \phi) = \rho_i(\phi)$, $1 \leq i \leq m$. For $1 \leq j \leq k$ define

$$\mathbb{W}_j(\phi) := \text{Span}\{e_i: \lambda(e_i, \phi) = \lambda_j(\phi)\}.$$

One then has $\dim \mathbb{W}_j(\phi) = m_j$ and $\mathbb{R}^m = \mathbb{W}_1(\phi) + \dots + \mathbb{W}_k(\phi)$.

The next lemma shows that every $y \in \mathbb{W}_j(\phi)$, $y \neq 0$, has $\lambda_j(\phi)$ as a strong Lyapunov exponent.

6.6. Lemma. Let $\phi \in H$ satisfy the hypothesis of Lemma 6.2. Then for
all $y \in W_j(\phi)$, $y \neq 0$, one has

$$(6.13) \quad \lim_{|t| \rightarrow +\infty} \frac{1}{t} \log |\Psi(\phi, t)y| = \lambda_j(\phi), \quad 1 \leq j \leq k.$$

Moreover the limit in (6.13) is uniform for $|y| = 1$.

Proof. We will use Lemma 6.3. Fix j and let $y \in W_j(\phi)$, $y \neq 0$. Then

$$y \in \text{Span}\{\text{col}_i(V) : \lambda_i(\phi) = \lambda_j(\phi)\}.$$

If $\{e_1, \dots, e_m\}$ is the natural basis in $\mathbb{R}^m$ then $z = V^{-1}y$ satisfies

$$z \in \text{Span}\{e_i : \lim_{|t| \rightarrow +\infty} \frac{1}{t} \log |\psi_{ii}(\phi, t)| = \lambda_j(\phi)\}.$$

Consequently one has

$$\lim_{|t| \rightarrow +\infty} \frac{1}{t} \log |D(\phi, t)z| = \lambda_j(\phi),$$

and the last limit is uniform for $|y| = 1$. Since one has

$$|S^{-1}(\phi, t)|^{-1} |D(\phi, t)z| \leq |\Psi(\phi, t)Vz| = |\Psi(\phi, t)y| \leq |S(\phi, t)| |D(\phi, t)z|$$

and

$$\liminf_{t \rightarrow +\infty} \frac{1}{t} \log |S^{-1}(\phi, t)|^{-1} = - \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |S^{-1}(\phi, t)| > 0$$

(by Lemma 6.3), we get

$$\begin{aligned} \lambda_j(\phi) &\leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log |S^{-1}(\phi, t)|^{-1} + \liminf_{t \rightarrow +\infty} \frac{1}{t} \log |D(\phi, t)z| \\ &\leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log |\Psi(\phi, t)y| \leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |\Psi(\phi, t)y| \\ &\leq \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |S(\phi, t)| + \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |D(\phi, t)y| \\ &\leq \lambda_j(\phi). \end{aligned}$$

A similar argument applies as $t \rightarrow -\infty$.

Q.E.D.

6.7 Lemma. Let $\phi \in H_v(\rho)$ and let $y \in \mathbb{R}^m$, $y \neq 0$. Assume that $\lambda(y, \phi) = \gamma$. Then there is a j , $1 \leq j \leq m$ such that $\lambda(y, \phi) = \lambda_j(\phi)$ and $y \in \mathbb{W}_j(\phi)$.

Proof. Since $e_1, \dots, e_m$ is a basis one has

$$(6.14) \quad y = \alpha_1 e_1 + \dots + \alpha_m e_m$$

where $\alpha_1, \dots, \alpha_m$ are scalars. It is an easy exercise to see that one has

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log |\Psi(\phi, t)y| = \max \{ \rho_i(\phi) : 1 \leq i \leq m \text{ and } \alpha_i \neq 0 \},$$

$$\lim_{t \rightarrow -\infty} \frac{1}{t} \log |\Psi(\phi, t)y| = \min \{ \rho_i(\phi) : 1 \leq i \leq m \text{ and } \alpha_i \neq 0 \}.$$

Therefore if the two-sided limit $\lambda(y, \phi) = \gamma$ exists, the only nonzero α 's in (6.14) are coefficients of basis vectors used to define a single $\mathbb{W}_j(\phi)$. Hence $\lambda(y, \phi) = \lambda_j(\phi)$ and $y \in \mathbb{W}_j(\phi)$. Q.E.D.

Let $\phi_1 = (\theta, U_1)$, $\phi_2 = (\theta, U_2)$ be two points in H with the same θ -coordinate. From (6.5) one has

$$(6.15) \quad G(\phi(\theta, t)U_1)\Psi(\phi_1, t)U_1^{-1} = \phi(\theta, t) = G(\phi(\theta, t)U_2)\Psi(\phi_2, t)U_2^{-1}.$$

Therefore if $y \in \mathbb{R}^m$ then $|\Psi(\phi_1, t)y| = |\Psi(\phi_2, t)Vy|$ where $V = U_2^{-1}U_1$. We see that

$$(6.16) \quad \lambda(y, \phi_1) = \gamma \iff \lambda(Vy, \phi_2) = \gamma.$$

Now assume further that $\phi_1 \in H_v$. Then ϕ_1 is an Lyapunov point by Lemma 6.2 and 6.4. Let $\gamma_1, \dots, \gamma_m$ be the strong Lyapunov exponents and let $e_1, \dots, e_m$ be a basis with $\lambda(e_i, \phi_1) = \gamma_i$, $1 \leq i \leq m$. It follows from (6.16) that $Ve_1, \dots, Ve_m$ is a basis for which $\lambda(Ve_i, \phi_2) = \gamma_i$, $1 \leq i \leq m$. We have just proved the following result:

6.8 Lemma. Let $\phi = (\theta, U) \in H_v$, and let $\gamma_1, \dots, \gamma_m$ be the set of strong Lyapunov exponents given by Lemma 6.2 and Eq. (6.4). Then every point $\hat{\phi} = (\theta, \hat{U})$ in the fibre over θ is a Lyapunov point with precisely the same set of strong Lyapunov exponents.

By combining Lemmas 6.7 and 6.8 and (6.15) we immediately have the following:

6.9 Lemma. Let $\phi_1 = (\theta, U_1)$ and $\phi_2 = (\theta, U_2)$ be two points in H_v with the same θ -coordinate. Then ϕ_1 and ϕ_2 lie in the same set $H_v(p)$, and for $1 \leq j \leq k$ one has

$$(6.17) \quad \lambda_j(\phi_1) = \lambda_j(\phi_2) \quad , \quad U_1 \mathbb{W}_j(\phi_1) = U_2 \mathbb{W}_j(\phi_2) .$$

Hence $\lambda_j(\phi_1)$ and $\mathbb{W}_j(\phi_1)$ depend only on the θ -coordinate.

By using (6.16) together with Lemma 6.6 we see that if $\phi_1 = (\theta, U_1) \in H_v$ then for any $\phi_2 = (\theta, U_2)$, with the same θ -coordinate, we have

$$\lambda(y, \phi_1) = \lambda(Vy, \phi_2) = \lambda_j(\phi_1)$$

for all $y \in \mathbb{W}_j(\phi_1)$, $y \neq 0$, where $V = U_2^{-1}U_1$.

We now use $r: H \rightarrow M$ to project H_v and $H_v(p)$ to M . Define M_μ and $M_\mu(p)$ by

$$r(H_v) := M_\mu \quad , \quad r(H_v(p)) := M_\mu(p) .$$

Note that since M and H are compact metric spaces, and H_v and $H_v(p)$ are Borel measurable sets in H , the images M_μ and $M_\mu(p)$ are μ -measurable sets in M , see Federer (1969, Chap. 2). Furthermore one has $\mu(M_\mu) = 1$.

(Strictly speaking, M_μ and $M_\mu(p)$ depend on the choice of $v \in I(\mu)$. Since $\mu(r(H_v)) = v(H_v) = 1$ we see that any two such sets M_μ agree except on a set of μ -measure 0.)

Let $\phi = (\theta, U) \in H_V(p)$. Then $\theta \in M_V(p)$. Next define

$$\lambda_j(\theta) := \lambda_j(\phi) \quad , \quad \mathbb{W}_j(\theta) := U(\mathbb{W}_j(\phi)) \quad , \quad 1 \leq j \leq k .$$

From Lemma 6.9 we see that $\lambda_j(\theta)$ and $U(\mathbb{W}_j(\phi))$ depend only on the θ -coordinate. Also from Lemmas 6.5 and 6.9 we see that $\lambda_1, \dots, \lambda_k: M_\mu(p) \rightarrow \mathbb{R}$ are μ -measurable and invariant. For $\theta \in M_\mu(p)$ we see that the spaces $\mathbb{W}_1(\theta), \dots, \mathbb{W}_k(\theta)$ satisfy conclusions (v)-(vii) of Theorem 2.1.

The only point that remains to be proven is that the mappings

$\mathbb{W}_i: M_\mu(p) \rightarrow \mathbb{G}(m, m_i)$, $1 \leq i \leq k$, are μ -measurable. Because of Lemma 6.9 it suffices to show that each $\mathbb{W}_i$ is Borel measurable on $H_V(p)$, $1 \leq i \leq k$. We will do this by noting that the basis matrix $V = \{e_1, \dots, e_m\}$ constructed in Eq. (6.7) is Borel measurable in ϕ since the coefficients in the integral depend continuously in ϕ , and therefore the integral is measurable* in ϕ , cf. Federer (1969). This completes the proof of Theorem 2.1.

Theorem 2.2, the Multiplicative Ergodic Theorem on a vector bundle $\mathbb{E}$, follows directly from Lemma 3.4 and Theorem 2.1. In Lemma 3.4 one can choose the $\lambda \in \mathbb{R}$ arbitrarily. A good choice for λ is $\lambda \notin \text{dyn}\Sigma(\mathbb{E})$, where $\text{dyn}\Sigma(\mathbb{E})$ is the dynamical spectrum of the linear skew product flow on $\mathbb{E}$. With this choice one knows that the measurable subbundle associated with λ is $\text{Range}(Q_2)$ and is disjoint from $\text{Range}(Q_1)$. (See Theorem 8.1 below.)

Remarks 6.1. The uniformity described in Lemma 6.6 can be strengthened. Let $\phi = (\theta, U) \in H_V(p)$, let $\lambda_1, \dots, \lambda_k: H_V(p) \rightarrow \mathbb{R}$ be the growth rates with $\lambda_1(\phi) < \dots < \lambda_k(\phi)$, and let

$$\mathbb{R}^m = \mathbb{W}_1(\phi) + \dots + \mathbb{W}_k(\phi)$$

* Discontinuities in ϕ can arise from the definition of τ in Lemma 6.2

be the decomposition of R^m into the measurable bundles. Then every $y \in R^m$ can be written uniquely as $y = y_1 + \dots + y_k$ where $y_i \in W_i(\phi)$, $1 \leq i \leq k$. Furthermore for $y \neq 0$ one has

$$(6.18) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \log |\Psi(\phi, t)y| = \lambda_b(\phi)$$

$$(6.19) \quad \lim_{t \rightarrow -\infty} \frac{1}{t} \log |\Psi(\phi, t)y| = \lambda_a(\phi)$$

where $a = \min\{i: y_i \neq 0\}$ and $b = \max\{i: y_i \neq 0\}$. (See Lemma 6.7.) By using the argument of Lemma 6.6, it is easily seen that the limits in (6.18-19) are uniform on compact sets of the form

$$\begin{aligned} &\{y \in R^m : 0 < \alpha < |y_b|, |y| < \beta\} \\ &\{y \in R^m : 0 < \alpha < |y_a|, |y| < \beta\}. \end{aligned}$$

These considerations extend immediately to the cocycle $\Phi(\theta, t)$ over M , where $W_i(\phi)$ is replaced by $\bigcup W_i(\phi)$, $1 \leq i \leq k$. (See Lemma 6.9)

6.2 As noted by Oseledec (1968) the uniformity condition in Lemma 6.6 implies that the limits

$$(6.20) \quad \lim_{|t| \rightarrow +\infty} \frac{1}{|t|} \log \beta_i(\theta, t), \quad 1 \leq i \leq m$$

exist almost everywhere, where $\beta_1 > \beta_2 > \dots > \beta_m$ are the eigenvalues of the positive self-adjoint matrix $\Phi^*(\theta, t)\Phi(\theta, t)$.

6.3 The basis $e_1, \dots, e_m$, which we construct in Lemma 6.2, is very closely related to Lyapunov's concept of "regularity" or "biregularity", see Lyapunov (1892) and Bylov, et al (1966). Note that if $\theta \in M_\mu$ then there are real numbers $\gamma_1 < \gamma_2 < \dots < \gamma_k$ and a splitting

$$R^m = W_1 + \dots + W_k$$

such that if $x \in W_i$, $x \neq 0$, then $\lambda(x, \theta) = \gamma_i$, $1 \leq i \leq k$, and

$$(6.21) \quad \sum_{i=1}^k m_i \gamma_i = \lim_{|t| \rightarrow +\infty} \frac{1}{t} \log |\det(\phi(\theta, t))| ,$$

where $m_i = \dim W_i$, $1 \leq i \leq k$. If ϕ is a smooth cocycle, i.e. if ϕ is a fundamental solution matrix of (6.1), then (6.21) becomes

$$\sum_{i=1}^k m_i \gamma_i = \lim_{|t| \rightarrow +\infty} \frac{1}{t} \int_0^t \text{tr } A(\theta \cdot s) ds .$$

Also see Vinograd (1956).

6.4 Other Proofs of the Multiplicative Ergodic Theorem. The proof of Oseledec (1968) uses many features of our argument, including the triangularization method described in Section IV and the theory of regularity described above. Some complication in Oseledec's argument seems to be due to the fact that he used neither Eq. (6.7) nor the factorization technique described in Lemmas 6.3 and 6.6. Also, Oseledec did not assume the base space M to be a compact metric space, and consequently his proof of the measurability (with respect to θ) of the bundles $\mathbb{W}_i(\theta)$ leaves some unanswered questions.

A portion of the Multiplicative Ergodic Theorem was derived by Millionscikov (1968) for the case where μ is an ergodic measure. He constructed the measurable spectrum $\text{meas}\Sigma(\mu)$ and showed that it was constant almost everywhere. Equation (6.7) was used by Millionscikov, however he did not derive Lemma 6.6, nor did he address the question of the measurability of the bundles $\mathbb{W}_i(\theta)$.

Raghunathan (1979), Ruelle (1979) and Crauel (1981) give alternate proofs of the Multiplicative Ergodic Theorem. Their approach is based on either a theorem of Furstenberg and Kesten (1960) (see Section X) or the Subadditive Ergodic Theorem, which was proved by Kingman (1968). Ruelle, for example, first shows that the limits in (6.20) exist almost everywhere. By using the eigenspaces of the associated self-adjoint operator $\phi^*(\theta, t)\phi(\theta, t)$, he constructs the measurable subbundles $\mathbb{W}_i(\theta)$.

The proof by Ruelle is more general than ours in that it applies to certain linearized semiflows generated by evolutionary equations on an infinite dimensional Hilbert space. Ruelle does not assume the base space M to be compact, instead he uses a logarithmic-boundedness condition on the cocycle Φ . This boundedness condition is automatically satisfied when the base space is compact. As we shall see in Section X, the assumption that M be compact is not a serious restriction, since this can be satisfied in practically every application.

VII. Flow on the Projective Bundle

In this section we shall study the ergodic properties of the induced flow on the projective bundle, see Johnson (1978, 1980b) and Crauel (1981).

As in Section VI, we let $\phi = (\theta, U) \in H_V(p)$ and let $\lambda_1(\phi) < \dots < \lambda_k(\phi)$ be the growth rates with multiplicity $\vec{m} = (m_1, \dots, m_k)$, where $m_1 + \dots + m_k = m$. By Lemma 6.9 we recall that $\lambda_i(\phi)$ depends only on θ , $1 \leq i \leq k$. Next define

$$U_i^\pm(\phi) = \text{Span}\{y \in \mathbb{R}^m : y \neq 0 \text{ and } \limsup_{t \rightarrow \pm\infty} \frac{1}{|t|} \log |\Psi(\phi, t)y| \leq \lambda_i(\phi)\}.$$

Then $\dim U_i^+(\phi) = m_1 + \dots + m_i$ and $\dim U_i^-(\phi) = m_i + \dots + m_k$. Also one has $\mathbb{W}_i(\phi) = U_i^+(\phi) \cap U_i^-(\phi)$, $\dim \mathbb{W}_i(\phi) = m_i$ and $\mathbb{R}^m = \mathbb{W}_1(\phi) + \dots + \mathbb{W}_k(\phi)$.

By Lemma 6.9 we see that $\mathbb{W}_i(\theta) := \bigcup \mathbb{W}_i(\phi)$ depends only on θ .

Let $\mathbb{P}^{m-1}(\mathbb{R})$ denote the projective space of lines in $\mathbb{R}^m$ containing the origin, with the usual topology. We define the induced flow $\hat{\pi}$ on the projective bundle $N = \mathbb{P}^{m-1}(\mathbb{R}) \times M$ by $(\ell, \theta) \cdot t = (\phi(\theta, t)\ell, \theta \cdot t)$. (Since ϕ is linear it maps lines onto lines.)

If $T = \mathbb{R}$, we define $f: N \rightarrow \mathbb{R}$ by $f(\ell, \theta) = \langle A(\theta)x, x \rangle$, where A is the matrix function (6.1), $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^m$, and $x \in \ell$ satisfies $|x| = 1$. Then for $|x| = 1$ one has

$$(7.1) \quad \log |\phi(\theta, t)x| = \int_0^t f((\ell, \theta) \cdot s) ds.$$

If $T = \mathbb{Z}$, we define $f: N \rightarrow \mathbb{R}$ by $f(\theta, \ell) = \frac{1}{2} \log \langle A^*(\theta)A(\theta)x, x \rangle$, where $A(\theta) = \phi(\theta, 1)$ and $|x| = 1$. Then one has

$$(7.2) \quad \begin{aligned} \log |\phi(\theta, t)x| &= \sum_{s=0}^{t-1} f((\ell, \theta) \cdot s) & t > 1; \\ \log |\phi(\theta, t)x| &= \sum_{s=1}^{|t|} f((\ell, \theta) \cdot -s) & t < 0. \end{aligned}$$

Next define the time-averages

$$f^+(\ell, \theta) = \limsup_{t \rightarrow +\infty} \frac{1}{t} \int_0^t f((\ell, \theta) \cdot s) ds ,$$

and

$$f^-(\ell, \theta) = \liminf_{t \rightarrow -\infty} \frac{1}{|t|} \int_0^t f((\ell, \theta) \cdot s) ds$$

when $T = \mathbb{R}$. (For $T = \mathbb{Z}$, $f^\pm$ are defined similarly by using (7.2).) Then by (7.1-2) $f^+(\ell, \theta)$ and $f^-(\ell, \theta)$ are the Lyapunov exponents $\lambda_S^+(x, \theta)$ and $\lambda_i^-(x, \theta)$, respectively, where $x \in \ell$, $x \neq 0$. Also the functions $f^\pm: \mathbb{N} \rightarrow \mathbb{R}$ are Borel measurable.

For $\theta \in M_\mu(p) = r(H_\nu(p))$ and $1 \leq i \leq k$ we define

$$\begin{aligned} u_i^+(\theta) &= \{\ell \in P^{m-1}(R) \mid f^+(\ell, \theta) < \lambda_i(\theta)\} , \\ u_i^-(\theta) &= \{\ell \in P^{m-1}(R) \mid f^-(\ell, \theta) < -\lambda_i(\theta)\} . \end{aligned}$$

For θ fixed, $u_i^\pm(\theta)$ are closed subsets of $P^{m-1}(R)$, and in fact are the "traces" in $P^{m-1}(R)$ of the vector subspaces $U_i^\pm(\theta)$ of R^m defined above. This leads us to introduce the space $\mathbb{K}$ of closed subsets of $P^{m-1}(R)$, with the Hausdorff topology. Thus $F_n \rightarrow F$ in $\mathbb{K} \iff$ to each $x \in F$, there corresponds a sequence $x_n \in F_n$ so that $x_n \rightarrow x$ in $P^{m-1}(R)$. Observe that the "trace" of $\mathbb{W}_i(\theta)$ in $P^{m-1}(R)$ is $u_i^+(\theta) \cap u_i^-(\theta)$.

Fix the pair $p = (k, m)$ and restrict attention to $M_\mu(p)$. The following proposition is a direct consequence of the measurability of the exponents $\lambda_1, \dots, \lambda_k$.

7.1 Lemma. For every $r > 0$, there is a compact set $Z \subseteq M_\mu(p)$ such that $\mu(M_\mu(p) \setminus Z) < r$ and the restriction $\lambda_i|_Z$ is continuous, $1 \leq i \leq k$.

We will now show that the functions $u_i^\pm$ are μ -measurable. (The measurability of $u_i^\pm$ is also a consequence of this fact.) Consider u_i^+ and $T = R$. (The arguments for u_i^- and $T = Z$ are similar and we will omit them.) Define $g_t(\ell, \theta) := \frac{1}{t} \int_0^t f((\ell, \theta) \cdot s) ds$. Then g_t is continuous on N , and $\limsup_{t \rightarrow +\infty} g_t(\ell, \theta_0) = f^+(\ell, \theta)$ for all $(\ell, \theta) \in \times P^{m-1}(R) \times M_\mu(p)$. Let $r > 0$ be given, and let $Z \subseteq M_\mu(p)$ be a compact set with $\mu(M_\mu(p) \setminus Z) < r$, where λ_i is continuous on Z , $1 \leq i \leq k$. Choose δ so that

$$(7.3) \quad 0 < 3\delta < |\lambda_i(\theta) - \lambda_j(\theta)|$$

for all $i \neq j$ and $\theta \in Z$. Finally define

$$v_t^+(\theta) := \{\ell \in P^{m-1}(R) \mid g_t(\ell, \theta) \in (-\infty, \lambda_i(\theta) + \delta]\}.$$

Then $v_t^+(\theta) \in \mathbb{K}$ for $\theta \in Z$. It is not difficult to verify that

$v_t^+ : Z \rightarrow \mathbb{K}$ is a Borel measurable function. (In general it is not continuous.)

We claim that $v_t^+(\theta) \rightarrow u_i^+(\theta)$ in $\mathbb{K}$ as $t \rightarrow +\infty$, for each $\theta \in Z$. Assume on the contrary that there is a monotone subsequence $t_k \rightarrow +\infty$ and an element $Q \in \mathbb{K}$ such that $Q \neq u_i^+(\theta)$ and $v_{t_k}^+(\theta) \rightarrow Q$. Then $u_i^+(\theta) \subseteq Q$, since $\ell \in u_i^+(\theta) \Rightarrow \ell \in v_{t_k}^+(\theta)$ for large t . On the other hand, let $\ell \in Q \setminus u_i^+(\theta)$. Since $v_{t_k}^+(\theta) \rightarrow Q$ in the Hausdorff topology, there is a sequence $\ell_k \in v_{t_k}^+(\theta)$ with $\ell_k \rightarrow \ell$. Therefore ℓ_k is eventually in every neighborhood of ℓ in $P^{m-1}(R)$. By Lemma 7.2 (below) we conclude that $g_{t_k}(\ell_k, \theta) > \lambda_i(\theta) + 2\delta$, which contradicts the definition of $v_{t_k}^+(\theta)$.

7.2 Lemma. Let $\ell \in \mathbb{P}^{m-1}(\mathbb{R})$ with $\ell \notin u_i^+(\theta)$ where $\theta \in Z$. Then given
any $\delta > 0$ there is a neighborhood $N(\ell)$ of ℓ and a $\tau \in T$ such that for all
 $t > \tau$ one has $g_t(\tilde{\ell}, \theta) > \lambda_i(\theta) + 2\delta$ for all $\ell \in N(\tilde{\ell})$.

Proof. We will use the notation of Remark 6.1. Let $x \in \ell$, $x \neq 0$. Since $\ell \notin u_i^+(\theta)$ one has $i < b$ where

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log |\Phi(\theta, t)x_b| = \lambda_b(\theta)$$

and $|x_b| = 2\alpha > 0$. Let $N(\ell)$ be those lines $\tilde{\ell}$ in $\mathbb{P}^{m-1}(\mathbb{R})$ with the property that $\tilde{x} \in \tilde{\ell}$, $|\tilde{x}| = 1$, satisfies $|\tilde{x}_b| > \alpha$. The uniformity assertion in Remark 6.1 implies that for every $\beta > 0$ there is a $\tau \in T$ such that

$$g_t(\tilde{\ell}, \theta) = \frac{1}{t} \log |\Phi(\theta, t)\tilde{x}| > \lambda_b(\theta) - \beta$$

for all $t > \tau$ and all $\tilde{x} \in \tilde{\ell} \in N(\ell)$ with $|\tilde{x}| = 1$. Now set $\beta = \delta$, then the lemma follows from (7.3). Q.E.D.

We see then that u_i^+ is the point-wise limit of a sequence of Borel measurable functions on Z . By the Lusin Theorem, it follows that u_i^+ is measurable on $M_\mu(p)$.

VIII. Comparison with the Continuous Spectrum

Let ϕ be a cocycle on a compact, connected Hausdorff space M . Let $\text{dyn}\Sigma = \bigcup_{i=1}^k [a_i, b_i]$ be the dynamical spectrum with the corresponding Whitney decomposition of $\mathbb{R}^m \times M$ into continuous spectral subbundles $\mathbb{R}^m \times M = \mathbb{V}_1 + \dots + \mathbb{V}_k$. Let $\mathbb{V}_i(\theta)$ denote the fibre of $\mathbb{V}_i$ in $\mathbb{R}^m$, $1 < i < k$. The following result is proved in Sacker and Sell (1978):

8.1 Theorem. The spectral subbundles $\mathbb{V}_i$ are characterized by

$$\mathbb{V}_i(\theta) = \text{Span}\{x \in \mathbb{R}^m : x \neq 0 \text{ and } \lambda_S^\pm(x, \theta), \lambda_i^\pm(x, \theta) \in [a_i, b_i]\}$$

for $1 < i < k$, where

$$\lambda_S^\pm(x, \theta) = \limsup_{t \rightarrow \pm\infty} \frac{1}{t} \log |\phi(\theta, t)x|, \quad \lambda_i^\pm(x, \theta) = \liminf_{t \rightarrow \pm\infty} \frac{1}{t} \log |\phi(\theta, t)x|.$$

We will next give a proof of Theorem 2.3. The essence of the argument is to verify (2.7), i.e.

$$\text{Boundary dyn}\Sigma \subseteq \bigcup_{\mu} \text{meas } \Sigma(\mu) \subseteq \text{dyn}\Sigma,$$

and to show that the measurable subbundle decomposition implied by the Multiplication Ergodic Theorem leads to a refinement of the continuous decomposition given by the Spectral Theorem.

It follows immediately from Theorem 8.1 that for any invariant probability measure μ on M one has $\text{meas}\Sigma(\mu, \theta) \subseteq \text{dyn}\Sigma$ for all $\theta \in M_\mu$. In particular one has $\text{meas}\Sigma(\mu) \subseteq \text{dyn}\Sigma$ for every ergodic measure μ on M . Furthermore the measurable bundles $\mathbb{W}_j(\theta)$ are contained in the associated continuous spectral bundle $\mathbb{V}_i(\theta)$ when $\lambda_j(\theta) \in [a_i, b_i]$, and $\theta \in M_\mu$. Since the sum of both the $\mathbb{W}_j(\theta)$'s and the $\mathbb{V}_i(\theta)$'s span $\mathbb{R}^m$ for $\theta \in M_\mu$, it follows that

for all $\theta \in M_\mu$ one has $\mathbb{V}_i(\theta) = \sum \mathbb{W}_j(\theta)$, where the summation is over all j with $\lambda_j(\theta) \in [a_i, b_i]$, $1 \leq i \leq k$. (This also follows from applying Theorem 2.2 to the spectral subbundle $\mathbb{V}_i$.)

It remains to show that if $\beta \in \text{Boundary dyn}\Sigma$ then $\beta \in \text{meas}\Sigma(\mu)$ for some ergodic measure μ . Let β be an endpoint of one of the spectral intervals $[a_i, b_i]$ in $\text{dyn}\Sigma$. Let $\mathbb{V}_i$ be the continuous spectral subbundle associated with $[a_i, b_i]$. As noted in Lemma 3.1 there is no loss in generality in assuming M to be a compact metric space. Let X_i be the trace of $\mathbb{V}_i$ in the projective bundle $N = P^{m-1}(R) \times M$, i.e.

$$X_i = \{(\ell, \theta) : \ell \text{ is a line in } \mathbb{V}_i(\theta)\}.$$

Since $\mathbb{V}_i$ is invariant under the flow $\pi(x, \theta, t) = (\phi(\theta, t)x, \theta \cdot t)$, X_i is invariant under the induced flow $\hat{\pi}$ on N . Also X_i is compact. Let $f(\ell, \theta)$ be given as in Section VII. Recall that the time-average of $f(\ell, \theta)$ along orbits in N determines the Lyapunov exponents of the solution $\phi(\theta, t)x$ where x is on the line ℓ , $x \neq 0$.

Let J be the set of all invariant measures η on X_i . We claim that

$$(8.1) \quad a_i < \int_{X_i} f d\eta < b_i$$

for all $\eta \in J$. If, on the contrary, (8.1) is false for some $\eta \in J$, then for η -almost all $(\ell, \theta) \in X_i$ one has

$$\lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\phi(\theta, t)x| = \hat{f}(\ell, \theta)$$

where f is an invariant function defined on N with $\int_{X_i} \hat{f} d\eta = \int_{X_i} f d\eta$, x is on ℓ , $x \neq 0$, and $\int_{X_i} f d\eta \notin [a_i, b_i]$. This implies that $\hat{f}(\ell, \theta) \notin [a_i, b_i]$ on some invariant set of positive η -measure, which contradicts Theorem 8.1.

Next we claim that there is a measure $\eta \in J$ such that $\int_{X_i} f d\eta = \beta$. To see this, assume for definiteness that $\beta = b_i$ is the right endpoint of $[a_i, b_i]$. Recall that J is compact, and that the mapping $\eta \mapsto \int_{X_j} f d\eta$ is continuous. Therefore if there is no $\eta \in J$ with $\int_{X_i} f d\eta = \beta$, then it follows from (8.1) that there is an $\epsilon > 0$ such that $\int_{X_j} f d\eta < \beta - \epsilon$ for all $\eta \in J$. It follows from the Krylov-Bogoliubov method described in Section V that for every $(x, \theta) \in \mathbb{Q}_i$ with $x \neq 0$ one has

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |\phi(\theta, t)x| = \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\pi(x, \theta, s)) ds < \beta - \epsilon.$$

It then follows from Sacker and Sell (1978, Lemma 4) that $\beta \notin \text{dyn}\Sigma$, a contradiction. A similar argument works for $\beta = a_i$.

We want to show next that η can be chosen to be an ergodic measure on X_i . Now fix $\eta \in J$ with $\int_{X_j} f d\eta = \beta$. Since X_j is a metric space, we can use the Choquet Representation Theorem, Phelps (1966), to find a probability measure A_η on the set of ergodic measures E in J such that

$$\int_{X_j} g d\eta = \int_E \left(\int_{X_j} g d\sigma \right) dA_\eta(\sigma)$$

for each $g \in \mathcal{C}(X_i, \mathbb{R})$. In particular for $f = g$ one has

$$\beta = \int_{X_j} f d\eta = \int_E \left(\int_{X_j} f d\sigma \right) dA_\eta(\sigma),$$

From (8.1) we see that one cannot have $\int_{X_j} f d\sigma < \beta = b_i$ for all $\sigma \in E$. It follows then that there is an ergodic measure $\sigma \in E$ with $\int_{X_j} f d\sigma = \beta$.

Finally let μ be the projection of σ to M . Then μ is an ergodic measure on M , and from Section VI we see that $\beta \in \text{meas}\Sigma(\mu)$. This completes the proof of Theorem 2.3. Q.E.D.

Remarks 8.1 In general, one cannot find a single ergodic measure μ such that $a_i, b_i \in \text{meas}\Sigma(\mu)$ for all endpoints a_i, b_i , even if M is minimal. Here is a simple example. According to Furstenberg (1961), there is a discrete flow on the 2-torus $M = T^2$ with more than one (in fact uncountably many) ergodic measures. Moreover, there is a continuous function g on M so that $\int_M g \, d\mu_1 \neq \int_M g \, d\mu_2$ for distinct ergodic measures μ_1, μ_2 . Define $\Phi : M \times \mathbb{Z} \rightarrow \mathbb{R}$ by $\Phi(y, 1) = \exp g(y)$. The dynamical spectrum of the cocycle Φ is $[a, b]$, where $a = \inf \int_M g \, d\mu$, $b = \sup \int_M g \, d\mu$, and the inf and sup are taken over all ergodic measures on M . However the measurable spectrum contains just the point $\{\int_M g \, d\mu\}$ for each ergodic measure μ .

8.2 If there is only one ergodic measure μ on M , for example if the flow $\theta \cdot t$ on M is almost periodic, then $\text{meas}\Sigma(\mu)$ is a subset of the $\text{dyn}\Sigma$ and all endpoints a_i, b_i of $\text{dyn}\Sigma$ are in $\text{meas}\Sigma(\mu)$. For $m = 2$ we conclude that $\text{meas}\Sigma(\mu) = \text{Boundary } \text{dyn}\Sigma$. An example in Johnson (1985) shows that for $m = 3$, even if M is almost periodic, the measurable spectrum need not consist entirely of endpoints a_i, b_i .

IX. Computation of the Measurable Spectrum. Wedge Product Flows

Let $\theta \in M$ be a Lyapunov point and let $\gamma_1(\theta) < \dots < \gamma_m(\theta)$ denote the growth rates with associated basis $e_1, \dots, e_m$. Thus one has $\lambda(e_i, \theta) = \gamma_i(\theta)$, $1 \leq i \leq m$. By a standard argument, see Naylor and Sell (1982, p. 268) for example, there is a constant K such that for any vector $x \in \mathbb{R}^m$ one has $x = \alpha_1 e_1 + \dots + \alpha_m e_m$ and

$$(9.1) \quad |\Phi(\theta, t)x| \leq K \max\{|\Phi(\theta, t)e_i| : \alpha_i \neq 0\} |x|$$

for all $t \in T$. It follows from (9.1) that

$$\limsup_{t \rightarrow +\infty} \frac{1}{t} \log |\Phi(\theta, t)| \leq \gamma_m(\theta).$$

On the other hand one has

$$|\Phi(\theta, t)e_m| \leq |\Phi(\theta, t)| |e_m|,$$

which implies that

$$\gamma_m(\theta) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log |\Phi(\theta, t)e_m| \leq \liminf_{t \rightarrow +\infty} \frac{1}{t} \log |\Phi(\theta, t)|,$$

and hence one has

$$(9.2) \quad \lim_{t \rightarrow +\infty} \frac{1}{t} \log |\Phi(\theta, t)| = \gamma_m(\theta).$$

A similar argument yields

$$(9.3) \quad \lim_{t \rightarrow -\infty} \frac{1}{t} \log |\Phi(\theta, t)| = \gamma_1(\theta).$$

An early version of (9.2) for stationary stochastic process of $(m \times m)$ matrices appears in Furstenberg and Kesten (1960).

The same considerations extend to the induced wedge product cocycles $\Lambda^k \Phi(\theta, t)$ on $\Lambda^k \mathbb{R}^m$, where $2 \leq k \leq m$. If $\Psi(\phi, t)$ satisfies (4.7), then one has

$$\Lambda^k \Psi(\phi, t) = (\Lambda^k q(\phi \cdot t))^{-1} (\Lambda^k \Phi(\theta, t)) (\Lambda^k q(\phi)).$$

Hence the cocycles $\Lambda^k \Psi$ and $\Lambda^k \Phi$ are cohomologous. Therefore $\Lambda^k \Psi$ and $\Lambda^k \Phi$ have the same Lyapunov exponents.

For $1 \leq k \leq m$ let $\text{Ord}(k, m)$ denote the collection of all strictly monotone mappings $\sigma: \{1, \dots, k\} \rightarrow \{1, \dots, m\}$. We will use the lexicographic ordering on $\text{Ord}(k, m)$, thus $\sigma < \tau$, where $\sigma, \tau \in \text{Ord}(k, m)$, provided there is a j , $1 \leq j \leq k$ such that $\sigma(i) = \tau(i)$ for $1 \leq i \leq j-1$ and $\sigma(j) < \tau(j)$.

If $\{e_1, \dots, e_m\}$ is any basis for R^m , then $\{e_\sigma : \sigma \in \text{Ord}(k, m)\}$ is a basis for $\Lambda^k R^m$ where

$$(9.4) \quad e_\sigma = e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(k)}.$$

Furthermore if T is an upper-triangular $(m \times m)$ matrix (with respect to the basis $\{e_1, \dots, e_m\}$), then $\Lambda^k T$ is an upper-triangular matrix with respect to the basis e_σ , $\sigma \in \text{Ord}(k, m)$. Also the diagonal entry $t_{\sigma\sigma}$ (in the σ -th position on the diagonal) is given by the product

$$t_{\sigma\sigma} = t_{\sigma(1)\sigma(1)} \cdot \dots \cdot t_{\sigma(k)\sigma(k)}.$$

If, in addition, one has $t_{ii} > 0$ for $1 \leq i \leq m$, then $t_{\sigma\sigma} > 0$ for all $\sigma \in \text{Ord}(k, m)$.

Let us return to the triangular cocycle $\Psi(\phi, t)$. It follows from the last paragraph that if $\phi = (\theta, U)$ is fixed and if the diagonal entries of $\Psi(\phi, t)$ satisfy (6.6), then the diagonal entry $\psi_{\sigma\sigma}$ of $\Lambda^k \Psi$ satisfies

$$(9.5) \quad \lim_{|t| \rightarrow \infty} \frac{1}{t} \log |\psi_{\sigma\sigma}(\phi, t)| = \gamma_{\sigma(1)} + \dots + \gamma_{\sigma(k)},$$

where $\sigma \in \text{Ord}(k, m)$ and $2 \leq k \leq m$. The collection of numbers given by (9.5), where σ varies over $\text{Ord}(k, m)$, represents the Lyapunov exponents of $\Lambda^k \Psi$.

This analysis applies for every $\phi \in H_v$, where H_v is given by Lemma 6.4.

For $\phi \in H_v(p)$ we shall re-write the growth rates in the form (2.8) where

$$(9.6) \quad \gamma_1(\phi) < \gamma_2(\phi) < \dots < \gamma_m(\phi).$$

It then follows from (9.5) that the largest growth rate for $\Lambda^k \Psi$ is

$$\gamma_{m+1-k}(\phi) + \dots + \gamma_m(\phi)$$

and the smallest is

$$\gamma_1(\phi) + \dots + \gamma_k(\phi).$$

The argument in the first paragraph in this section now applies to $\Lambda^k \Psi$, which completes the proof of Theorem 2.4.

Remark 9.1 One can give a precise description of the measurable bundles $W_{\tau}^{(k)}(\phi)$ corresponding to $\Lambda^k \Psi$. Fix $\phi \in H_V(p)$, where the growth rates of Ψ satisfy (9.6), and let $e_1(\phi), \dots, e_m(\phi)$ be a basis in R^m that satisfies $\lambda(e_i(\phi), \phi) = \gamma_i(\phi)$, $1 \leq i \leq m$. For $\tau \in \text{Ord}(k, m)$ we define

$$\gamma_{\tau}(\phi) = \gamma_{\tau(1)}(\phi) + \dots + \gamma_{\tau(k)}(\phi).$$

Then one has

$$(9.7) \quad W_{\tau}^{(k)}(\phi) = \text{span} \{e_{\sigma} : \lambda(e_{\sigma}, \phi) = \gamma_{\tau}(\phi)\},$$

where e_{σ} is defined by (9.4) for $\sigma \in \text{Ord}(k, m)$. We will omit the proof of (9.7), which is a simple application of the techniques developed in Section VI.

X. Applications and Illustrations.

In this section we collect several illustrative examples of the theory described above. Included here is a discussion of spiral systems, products of "random" matrices, conservative second-order Schroedinger equations with almost periodic potentials and linear stochastic differential equations with bounded measurable coefficients.

(A) Spiral Systems. The theory above applies to every compact invariant set N in M . In this case the dynamical spectrum $\text{dyn}\Sigma(N)$ depends on N . Next we want to study the case where N is a single orbit together with its ω -limit set, i.e. a spiral system. More precisely let M be a compact Hausdorff space with a flow $\theta \cdot t$. Let θ_0 be a given point in M and define

$$N = H^+(\theta_0) = \text{Closure } \{\theta_0 \cdot t : t \geq 0\}.$$

Then N is positively invariant and the ω -limit set $\Omega = \bigcap_{\tau \geq 0} H^+(\theta_0 \cdot \tau)$ is a compact invariant set. We are interested in the case where $\theta_0 \notin \Omega$. Thus the positive trajectory $\theta_0 \cdot t$ forms a spiral. See Figure 1.

Figure 1

N : A spiral system

Let $\phi(\theta, t)$ be a cocycle defined on M . Then the theory described above applied directly to the restriction of ϕ to the ω -limit set Ω . The problem we wish to study here is the limiting behavior (as $s, \tau \rightarrow +\infty$) of the cocycles

$\Lambda^k \phi(\theta_0 \cdot s, t)$, $1 \leq k \leq m$, along the spiral trajectory θ_0 . In particular we want to show that this limiting behavior can be used to evaluate the measurable spectrum of the ω -limit set Ω .

Before doing the analysis it should be noted that this study addresses a basic question which arises naturally when one is doing a numerical evaluation of the measurable (or dynamical) spectrum. The initial point θ_0 in M is determined by the code or program. If by good fortune, it happens to lie in the set M_μ (Theorem 2.1) then Theorem 2.4 explains how to compute the spectrum. With our present understanding, one does not know whether or not we have had good fortune. However, what is always true is that θ_0 does determine a spiral system.

For $k = 1, \dots, m$ we define

$$b^k := \limsup_{s, \tau \rightarrow +\infty} \frac{1}{\tau} \log |\Lambda^k \phi(\theta_0 \cdot s, \tau)|.$$

Also let Σ^k denote the dynamical spectrum of the linear skew-product flows $(\Lambda^k \phi(\theta, t), \theta \cdot t)$ over Ω , and define

$$a^k := \max \Sigma^k.$$

We will now prove the following result:

10.1 Theorem. The following statements are valid:

(A) For $k=1, \dots, m$ one has $a^k = b^k$.

(B) If the flow on Ω is uniquely ergodic (e.g. almost periodic) then

$\text{meas} \Sigma = \{\gamma_1, \dots, \gamma_m\}$ where $\gamma_m = b^1$ and

$$\gamma_{m-k} = b^{k+1} - (\gamma_m + \dots + \gamma_{m-k+1}), \quad 1 \leq k \leq m-1.$$

Proof. We will give the proof of (A) for $k=1$. The argument for $k \geq 2$ is the same. Statement (B) is an immediate corollary of part (A) and Theorems 2.1, 2.3 and 2.4.

Let a and b denote a^1 and b^1 . For $S > 0$ and $T > 0$ we define

$$\beta(S, T) = \sup_{\substack{S \leq s \\ T \leq \tau}} \frac{1}{\tau} \log |\Phi(\theta_0 \cdot s, \tau)|.$$

Then $b = \lim_{S, T \rightarrow +\infty} \beta(S, T)$. Fix $\epsilon > 0$ and choose $S > 0, T > 0$ so that $\beta(S, T) < b + \epsilon$. For $t = s + \tau$ one then has

$$|\Phi(\theta_0 \cdot s, \tau)| = |\Phi(\theta_0, t) \Phi^{-1}(\theta_0, s)| < K e^{(b+\epsilon)(t-s)}, \quad 0 < s < t$$

where $K = \max \{|\Phi(\theta_0 \cdot s, \tau)| : 0 < s < S, 0 < \tau < T\}$. It then follows directly, see Sacker and Sell (1974, Theorems 2 and 5) and (1976a, Lemma 4) for example, that $a < b + \epsilon$ for every $\epsilon > 0$. Hence $a < b$.

If one has $a < b$, then we can replace $\Phi(\theta, t)$ by the shifted flow $e^{\lambda t} \Phi(\theta, t)$ where $3\lambda = 2a + b$. This has the effect of shifting a and b to $a - \lambda$ and $b - \lambda$, respectively. Without any loss of generality, therefore, we can assume then that $a < 0 < b$ and set $b = 3\alpha$. Since $a < 0$, the linear skew-product flow $(\Phi(\theta, t)x, \theta \cdot t)$ has an exponential dichotomy over Ω with $\mathbb{S} = \mathbb{R}^m \times \Omega$. This means that there is a constant K such that

$$|\Phi(\theta, t) \Phi^{-1}(\theta, s)| < K e^{-\alpha(t-s)}$$

for all $\theta \in \Omega$, and $s < t, s, t \in T$. In particular it follows from Sacker and Sell (1974, p. 452) that $\mathbb{B} = \{0\} \times \Omega$, $\mathbb{S} = \mathbb{R}^m \times \Omega$ and $\mathbb{U} = \{0\} \times \Omega$, where $\mathbb{B}$, $\mathbb{S}$ and $\mathbb{U}$ are defined to be those $(x, \theta) \in \mathbb{R}^m \times \Omega$ that satisfy

$$\sup_{t \in T} |\Phi(\theta, t)x| < \infty, \quad \lim_{t \rightarrow +\infty} |\Phi(\theta, t)x| = 0, \quad \text{and} \quad \lim_{t \rightarrow -\infty} |\Phi(\theta, t)x| = 0,$$

respectively.

Since $b > 0$ there are sequences $s_n \rightarrow +\infty, \tau_n \rightarrow +\infty$ such that

$$|\Phi(\theta_0 \cdot s_n, \tau_n)| > e^{\alpha \tau_n}.$$

Let e_n be a vector with $|e_n| = 1$ and

$$|\phi(\theta_0 \cdot s_n, \tau_n) e_n| > e^{\alpha \tau_n}.$$

Fix σ_n so that $0 < \sigma_n < \tau_n$ and

$$|\phi(\theta_0 \cdot s_n, t) e_n| < |\phi(\theta_0 \cdot s_n, \sigma_n) e_n|, \quad 0 < t < \tau_n.$$

Set $\xi_n = \phi(\theta_0 \cdot s_n, \sigma_n) e_n$. Then $|\xi_n| > e^{\alpha \tau_n}$, and

$$|\xi_n|^{-1} |\phi(\theta_0 \cdot s_n, t) e_n| < 1, \quad 0 < t < \sigma_n.$$

Since $|\xi_n| \rightarrow +\infty$ one has $\sigma_n \rightarrow +\infty$. Also one has $e_n = \phi(\theta_n, -\sigma_n) \xi_n$ where $\theta_n = \theta_0 \cdot (s_n + \sigma_n)$. By choosing subsequences (if necessary) we can assume that $\theta_n \rightarrow \theta \in \Omega$ and $|\xi_n|^{-1} \xi_n \rightarrow e$ where $|e| = 1$. It follows from Sacker and Sell (1976a, Lemma 4) that $(e, \theta) \in \mathbb{U}$, which contradicts the fact that $\mathbb{U} = \{0\} \times \Omega$. We conclude that $a = b$.

Remarks 10.1 The conclusions of Theorem 10.1 can be reformulated in another manner. The numbers a^k represent the "largest possible" Lyapunov exponents for the cocycle $\Lambda^k \phi(\theta, t)$, where $\theta \in \Omega$ and $1 \leq k \leq n$. By Ineq. (2.7) and Theorem 2.4 we see that there are ergodic measures μ_k on Ω with the property that a^k is the largest value in the measurable spectrum generated by $\Lambda^k \phi$, $1 \leq k \leq n$. Consequently if μ is any invariant measure on Ω and $\gamma_1(\theta) < \dots < \gamma_m(\theta)$ are the growth rates satisfying (9.6) for $\theta \in \Omega_\mu$, then one has

$$\gamma_m(\theta) + \dots + \gamma_{m-k+1}(\theta) < a^k$$

for all $\theta \in \Omega_\mu$.

10.2 The ergodic measures μ_k referred to in the last paragraph need not be the same. Let Ω be a minimal set in the flow $\theta \cdot t$, and assume that this

flow is not uniquely ergodic. Then there are ergodic measures μ_1 and μ_2 on Ω and a continuous real-valued function g that satisfies $\int g d\mu_1 \neq \int g d\mu_2$. Since g can be replaced by $g + c$ and/or αg , where c and α are constants with $\alpha \neq 0$, we can assume that μ_1 , μ_2 and g are chosen to satisfy

$$-1 = \int g d\mu_2 < \int g d\mu < \int g d\mu_1 = 3$$

for every ergodic measure μ . Now set $h = -2g$ and consider the linear skew-product flow on $\mathbb{R}^2 \times \Omega$ generated by

$$x' = \text{diag}(g(\theta \cdot t), h(\theta \cdot t))x.$$

In the notation introduced above one then has $a^1 = 3$, $a^2 = 2$, and $a^i = \max \text{meas } \sum (\mu_j)$, $i = 1, 2$.

10.3 It may happen that all the limits

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log |\Lambda^k \Phi(\theta_0, t)| = c^k, \quad 1 \leq k \leq m,$$

exist. If so, then one has $c^k \leq a^k$, $1 \leq k \leq m$. By using the associated triangular flow and the Krylov-Bogoliubov method described in Section V, one can show that there is an invariant measure μ on Ω with the property that $\text{meas } \sum (\mu) = \{\gamma_1, \dots, \gamma_m\}$ where $\gamma_1 \leq \dots \leq \gamma_m$ and

$$\gamma_{m-k+1} + \dots + \gamma_m = c^k, \quad 1 \leq k \leq m.$$

10.4 The conclusions of Theorem 10.1 and the above Remarks are related to results of Pelikan (1983) who has analyzed the structure of Bowen-Ruelle invariant measures on certain attractors.

(B) Products of Random Matrices. In this example we show that the theory of products of matrices considered in Furstenberg and Kesten (1960) is often-times imbeddable in our theory. Let K denote a fixed compact subset of $\textcircled{GL}(m)$ and let $\{X_1, X_2, \dots\}$ be a given sequence of $(m \times m)$ matrices with values in K .

For $n = 1, 2, \dots$ form the product

$$Y_n = X_n X_{n-1} \dots X_1.$$

We wish to study the limiting behavior of

$$\frac{1}{n} \log |Y_n|$$

as $n \rightarrow \infty$. This will be done by imbedding this problem into a spiral system, where $Y_n = \Phi(\theta_0, n)$ for an appropriate cocycle Φ , and using Theorem 10.1.

Let $M := K^{\mathbb{Z}}$ denote the collection of all two-sided sequences

$$(10.1) \quad \theta = (\dots, A_{-2}, A_{-1}; A_0; A_1, A_2, \dots)$$

with entries $A_i \in K$, $i \in \mathbb{Z}$. (We will use semi-colons to designate the zeroth position of θ .) Then M is a compact metric space with the shift flow $\theta \cdot n$ where

$$\theta \cdot n = (\dots, A_{n-2}, A_{n-1}; A_n; A_{n+1}, A_{n+2}, \dots)$$

for $n \in \mathbb{Z}$. Define $F: M \rightarrow K$ by $F(\theta) := A_0$ where θ is given by (10.1).

Then one constructs a cocycle Φ over M by defining $\Phi(\theta, 0) = 1$

$$\Phi(\theta, n) = F(\theta \cdot (n-1)) \dots F(\theta), \quad n > 1$$

$$\Phi(\theta, n) = [F(\theta \cdot (-1)) \dots F(\theta \cdot n)]^{-1}, \quad n < -1.$$

It is not difficult to see that the cocycle identity (2.1) is valid for $t, s \in \mathbb{Z}$.

The distinguished sequence $\{X_1, X_2, \dots\}$ is imbedded into this flow as a spiral, i.e. let A be a fixed element in K and set

$$\theta_0 = (\dots, A, A; X_1; X_2, X_3, \dots),$$

where every negative entry in θ_0 is A . Then $\theta_0 \in M$ and $Y_n = \Phi(\theta_0, n)$ for $n > 1$. By Theorem 10.1 we see that

$$(10.2) \quad \limsup_{m,n \rightarrow +\infty} \frac{1}{m} \log |\Phi(\theta \cdot n, m)| = \limsup_{m,n \rightarrow +\infty} \frac{1}{m} \log |Y_{m+n} Y_n^{-1}|$$

exists and this is the maximum value of the dynamical spectrum over Ω , the ω -limit set of θ_0 .

When the distinguished sequence $\{X_1, X_2, \dots\}$ is a stationary stochastic process, then the expectation satisfies $E(\log^+ |X_1|) < \infty$ since X_1 assumes values in the compact set K . If, in addition, this stochastic process is metrically transitive (i.e. ergodic) then Theorem 2.4 is applicable, and one concludes that

$$(10.3) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log |Y_n|$$

exists with probability 1. Also the limits in (10.2) and (10.3) agree. We refer the reader to Furstenberg and Kesten (1960) for more details.

Remarks 10.5 Some interesting applications of products of random matrices to problems in demographics can be found in Cohen (1979).

(C) Schroedinger Equation. In the study of the Schroedinger equation

$$Ly = \left(-\frac{d^2}{dt^2} + q(t)\right)y = \lambda y,$$

where $q(t)$ is real and Bohr almost periodic, it is of interest to compute the "Lyapunov number" $\beta(\lambda)$ as a function of $\lambda \in \mathbb{R}$. The definition of $\beta(\lambda)$ goes as follows: First introduce the hull M of q by

$$M = \text{Closure}\{q_\tau \mid \tau \in \mathbb{R}\},$$

where $q_\tau(t) = q(t + \tau)$, and the closure is in the uniform topology. Then M is a compact metric space with translation flow $\theta \cdot \tau = \theta_\tau$. In fact M is a compact topological group, and the normalized Haar measure μ is the unique

invariant measure on M . Define $Q(\theta) = \theta(0)$, and consider the operators

$L_\theta = -\frac{d^2}{dt^2} + Q(\theta \cdot t)$ and the associated equations

$$(10.4) \quad x' = \begin{pmatrix} 0 & 1 \\ -\lambda + Q(\theta \cdot t) & 0 \end{pmatrix} x, \quad x = \begin{pmatrix} y \\ y' \end{pmatrix}.$$

Since the trace of the coefficient matrix is zero, one obtains from Liouville's formula that

$$\text{meas} \Sigma(\mu) = \{-\beta(\lambda), \beta(\lambda)\}$$

where $\beta(\lambda) \geq 0$. This defines $\beta(\lambda)$. Also, as noted in Remark 8.2, one has $\text{Boundary dyn} \Sigma = \text{meas} \Sigma(\mu)$.

Spectral properties of the self-adjoint linear operators L_θ on $L^2(-\infty, \infty)$ are reflected in the dynamics of (10.4). For example λ is in the resolvent set for L_θ if and only if (10.4) admits an exponential dichotomy, Johnson (1982). Also if $\beta(\lambda) > 0$ for all λ in an interval I , then for μ -almost all θ , the (functional analytic) spectrum of L_θ has no absolutely continuous component in I , Pastur (1980), Ishii (1971). Moreover if $\beta(\lambda) = 0$ for all λ in I , then I is in the purely absolutely continuous spectrum of L_θ for μ -almost all θ , Kotani (1982).

The numerical computation of $\beta(\lambda)$ when $q(t) = \cos t + \cos \pi t$, for example, is a challenging problem. An investigation of this problem is described in Perry (1985). The basic idea is to use Theorem 2.4 to estimate $\beta(\lambda)$. Also special properties of second order linear equations, as described in Johnson (1980a) and Johnson and Moser (1982), can be exploited to help determine whether or not (10.4) admits an exponential dichotomy for $\lambda=0$. This, in turn, leads to a resolution of the question of whether or not one has $\text{dyn} \Sigma = \text{meas} \Sigma(\mu)$.

Another method for computing $\beta(\lambda)$, which was suggested by R. Helleman, is to use the theory of Johnson and Moser (1982). In this setting one extends λ to the complex plane so that for $\text{Im } \lambda > 0$, $\beta(\lambda)$ is the real part of a holomorphic function $w(\lambda)$, called the Floquet exponent of (10.4). When $\text{Im } \lambda > 0$, Eq. (10.4) has an exponential dichotomy. One can compute $\beta(\lambda)$ for real λ by a limiting formula: $\beta(\lambda) = \lim_{n \rightarrow 0^+} \beta(\lambda + in)$.

(D) Linear Stochastic Differential Equations. An interesting variation of the Schroedinger equation occurs when the potential $q(t)$ is a stochastic variable. More generally consider the m -dimensional case $x' = A(t)x$, $x \in \mathbb{R}^m$, where the entries $a_{ij}(t)$ are stochastic variables in t . We will show how this can be imbedded in a linear skew-product flow on $\mathbb{R}^m \times M$, where M is a compact space, under the assumption that coefficients $a_{ij}(t)$ are bounded and measurable in t , i.e. $a_{ij} \in L^\infty(\mathbb{R})$. (See Kurzweil (1957), Miller and Sell (1970) and Sacker and Sell (1974) for more information.)

The set M is the hull of A and is defined by

$$M = \text{Closure } \{A_\tau : \tau \in \mathbb{R}\},$$

where $A_\tau(t) = A(\tau + t)$ and the closure is taken in the "weak L^1 -local" topology. That is, a generalized sequence $\{A_n\}$ converges to a limit B if for every $\tau \in \mathbb{R}$ and every $\phi \in L^1[\tau, \tau+1]$ one has

$$\int_{\tau}^{\tau+1} A_n(t)\phi(t)dt \rightarrow \int_{\tau}^{\tau+1} B(t)\phi(t)dt.$$

Since $A(t)$ is bounded and measurable, the hull of A is a compact Hausdorff space. If $B \in M$ we let $\phi(B, t)$ be the fundamental solution matrix for $x' = B(t)x$. Then $\phi: M \times \mathbb{R} \rightarrow \text{GL}(m)$ is continuous and the associated linear skew-product flow is

$$\pi(B, x, \tau) = (\phi(B, \tau)x, B_\tau).$$

When the coefficients a_{ij} are stationary stochastic variables, it is not difficult to show (by using the ideas of Section V) that a given underlying invariant probability measure μ lifts to an invariant measure ν on M . If the coefficients are metrically transitive, i.e. if μ is ergodic, then the lifted measure ν can be chosen to be ergodic.

APPENDIX. Further Geometric Properties of Cocycles.

A projector is a continuous mapping $P(x, \theta) = (P(\theta)x, \theta)$ on $R^m \times M$ where M is a compact Hausdorff space and $P(\theta)$ is a linear projection on R^m . A resolution of the identity on $R^m \times M$ is a k -tuple $P = (P_1, \dots, P_k)$, $k \geq 1$, satisfying: (i) each P_i is a projector, (ii) $P_i P_j = \vec{0}$ when $i \neq j$ and (iii) $I = P_1 + \dots + P_k$. (Here we define $\vec{0}(x, \theta) := (0, \theta)$.) Let $P = (P_1, \dots, P_k)$ be a k -tuple of projectors on $R^m \times M$ and define

$$\mathbb{R}_i := \text{Range } (P_i) := \{(x, \theta) \in R^m \times M : P_i(\theta)x = x\}, \quad 1 \leq i \leq k.$$

Then P is a resolution of the identity if and only if $R^m \times M = \mathbb{R}_1 + \dots + \mathbb{R}_k$ (as a Whitney sum). A resolution of the identity P is said to be orthogonal if $\mathbb{R}_i \perp \mathbb{R}_j$ whenever $i \neq j$, i.e. the Euclidean inner product $\langle \cdot, \cdot \rangle$ satisfies $\langle x, y \rangle = 0$ for all $(x, \theta) \in \mathbb{R}_i$, $(y, \theta) \in \mathbb{R}_j$ when $i \neq j$. The latter is equivalent to saying that each $P_i(\theta)$ is an orthogonal projection on R^m .

Now let ϕ be a cocycle on $R^m \times M$ and assume $R^m \times M = \mathbb{V}_1 + \dots + \mathbb{V}_k$ as a Whitney sum. Let $P = (P_1, \dots, P_k)$ be the induced resolution of the identity where $\text{Range } (P_i) = \mathbb{V}_i$, $1 \leq i \leq k$. Then the subbundles $\mathbb{V}_i$ are invariant under the linear skew product flow induced by ϕ if and only if one has

$$(A.1) \quad P_i(\theta \cdot t) \phi(\theta, t) = \phi(\theta, t) P_i(\theta), \quad 1 \leq i \leq k$$

for all $\theta \in M$ and $t \in T$. We shall say that a resolution of the identity $P = (P_1, \dots, P_k)$ is invariant if (A.1) is satisfied. It is not always the case that an invariant resolution of the identity is orthogonal, however the next lemma shows that one can replace ϕ with a cohomologous flow in which the new invariant resolution of the identity is orthogonal.

Lemma A. Let ϕ be an cocycle over a compact Hausdorff space M and let $P = (P_1, \dots, P_k)$ be an invariant resolution of the identity. Then there is continuous self-adjoint mapping $R : M \rightarrow GL(m)$ with the property that $Q = (Q_1, \dots, Q_k)$ is an orthogonal resolution of the identity, where

$$(A.2) \quad Q_i(\theta) = R(\theta)P_i(\theta)R^{-1}(\theta) \quad , \quad 1 \leq i \leq k .$$

Furthermore Q is invariant under the cocycle

$$(A.3) \quad \psi(\theta, t) = R(\theta \cdot t)\phi(\theta, t)R^{-1}(\theta) .$$

Proof. Define $S(\theta)$ by

$$S(\theta) := \sum_{i=1}^k P_i^*(\theta)P_i(\theta)$$

where P^* denotes the adjoint operation. Then $S(\theta)$ is positive definite and self-adjoint, so it has a unique positive definite, self-adjoint square root $R(\theta)$, i.e. $R^2(\theta) = S(\theta)$. If $Q_i(\theta)$ is defined by (A.2) and ψ is given by (A.3) it is easy to verify that $Q_i^*(\theta) = Q_i(\theta)$ and $\psi(\theta, t)Q_i(\theta) = Q_i(\theta \cdot t)\psi(\theta, t)$, $1 \leq i \leq k$. Q.E.D.

Lemma 3.4 also gives information about the case where one has a linear skew product flow on a vector bundle $\mathbb{E}$ over a compact Hausdorff space M where $\mathbb{E} = \mathbb{V}_1 + \dots + \mathbb{V}_k$ is a Whitney sum of invariant subbundles. Each of these subbundles $\mathbb{V}_i$ can be separately imbedded in a trivial bundle $R^{m_i} \times M$ where $m = m_1 + \dots + m_k$. The construction of Lemma 3.4 shows that one can construct a cocycle on $R^m \times M$ so that the given flow on $\mathbb{E}$ is cohomologous to a flow on a subbundle of $R^m \times M$.

If π is a discrete flow on a vector bundle $\mathbb{E}$, i.e. if $T = Z$, then Lemma 3.4 can be extended to this case by first suspending the discrete flow to get an equivalent continuous-time flow on a new vector bundle. See Ellis and

Johnson (1982) for the suspension construction. One should note that even if the original bundle is trivial, the suspended bundle may be nontrivial.

The triangularization technique can be used to put some cocycles into a block-diagonal, upper-triangular form. Let ϕ be a cocycle on M and let $P = (P_1, \dots, P_k)$ be an invariant partition of unity of $R^m \times M$. Because of Lemma A we can assume P to be orthogonal. For any point $\phi = (\theta, U) \in H$ we define the P -partition of U to be the partitioning of U into block matrices $U = (U_1, \dots, U_k)_P$ where the number m_i of column vectors in U_i is $\dim \text{Range } P_i(\theta)$, $1 \leq i \leq k$. Let H_P denote the set of all $\phi = (\theta, U) \in H$ with the property that the P -partition $U = (U_1, \dots, U_k)_P$ satisfies

$$(A.4) \quad P_i(\theta)U_j = \delta_{ij}U_i, \quad 1 \leq i, j \leq k.$$

By using (A.2) one can easily verify the following:

Lemma B. H_P is a compact invariant set in H in the flow $\phi \cdot t$. Furthermore if $\phi = (\theta, U) \in H_P$ then the column vectors in $\phi(\theta, t)U_i$ are orthogonal to those in $\phi(\theta, t)U_j$ when $i \neq j$.

For $\phi = (\theta, U) \in H_P$ let $T(\phi(\theta, t)U)$ be given by (4.5). The P -partition of U prescribes an induced block partition of $T(\phi(\theta, t)U)$ where the diagonal blocks are square matrices of size $(m_i \times m_i)$, $1 \leq i \leq k$. Since the off-diagonal blocks of $T(\phi(\theta, t)U)$ depend on the inner products of column vectors of $\phi(\theta, t)U_i$ and $\phi(\theta, t)U_j$ for $i \neq j$, it follows from Lemma B that these off-diagonal blocks are zero. Hence $\Psi(\phi, t) = T(\phi(\theta, t)U)^{-1}$ is a block-diagonal, upper-triangular matrix.

The block-diagonalization of Ψ involves an "untwisting" of the spectral subbundles of Φ . A similar untwisting with additional useful structures appears in Ellis and Johnson (1982). Also compare with Coppel (1967), Palmer (1980), and Vinograd, et al (1977).

REFERENCES

1. M. Atiyah (1967) K-Theory, Benjamin, New York.
2. Bylov, et al (1966) The Theory of Lyapunov Exponents and Its Applications to Stability Problems (Russian), Nauka, Moscow.
3. A. Chenciner and G. Iooss (1979) Bifurcations de tores invariant, Arch. Ratl. Mech. Anal. 69, 109-198.
4. J.E. Cohen (1979) Ergodic theorems in demography, Bull. Amer. Math. Soc. (N.S.) 1, 275-295.
5. P. Constantin and C. Foias (1983) Global Lyapunov exponents, Kaplan - Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equations, IMA Preprint #21.
6. W.A. Coppel, (1978) Dichotomies in Stability Theory, Lecture Notes in Math., No. 629, Springer-Verlag.
7. H. Crauel (1981) Ergodentheorie linearer stochastischer systeme, Universitat Bremen, Report No. 59.
8. J. Daletskii and M. Krein (1974) Stability of Solutions of Differential Equations in Banach Space, AMS Translations, American Math. Society, Providence.
9. S.P. Diliberto (1950) On systems of ordinary differential equations, Contributions Thy. Nonlin. Oscil., Ann. Math. Studies no. 20, pp. 1-38.
10. R. Ellis (1969) Lectures on Topological Dynamics, Benjamin, New York.
11. R. Ellis and R. Johnson (1982) Topological dynamics and linear differential systems, J. Diff. Equations., 44, 21-39.
12. H. Federer (1969) Geometric Measure Theory, Springer-Verlag, New York.
13. N. Fenichel (1971) Persistence and smoothness of invariant manifolds for flows, Indiana Univ. Math. J., 21, 193-226.
14. C. Foias, G.R. Sell, and R. Temam (1985) Variete inertielles des equations differentielles dissipative, (to appear).
15. H. Furstenberg (1961) Strict ergodicity and transformations of the torus, Amer. J. Math. 83, 573-601.
16. H. Furstenberg and H. Kesten (1960) Products of random matrices, Ann. Math. Stat. 31, 457-469.
17. V. Guillemin and A. Pollack (1974) Differential Topology, Prentice-Hall, Englewood Cliffs.

18. E. Hewitt and K. Ross (1963) Abstract Harmonic Analysis I, Springer-Verlag, New York.
19. W.M. Hirsch, C.C. Pugh, and M. Shub (1977) Invariant Manifolds, Lecture Notes in Math., No. 583, Springer-Verlag.
20. K. Ishii (1973) Localization of eigenstates and transport phenomena in one-dimensional disordered systems, Supp. Theor. Phys. 53, 77-138.
21. R.A. Johnson (1978) Ergodic theory and linear differential equations, J. Diff. Equations 28, 23-34.
22. R.A. Johnson (1980a) On a Floquet theory for almost-periodic, two dimensional linear systems, J. Diff. Equations, 37, 184-205.
23. R.A. Johnson (1980b) Analyticity of spectral subbundles, J. Diff. Equations 35, 366-387.
24. R.A. Johnson (1982) On the recurrent Hill equation, J. Diff. Equations 46, 165-194.
25. R.A. Johnson (1985) An example of a three-dimensional linear differential system with almost-periodic coefficients, in preparation.
26. R.A. Johnson and J. Moser (1982) The rotation number for almost periodic potentials, Comm. Math. Phys. 84, 403-438.
27. A. Katok (1980) Lyapunov exponents, entropy and periodic orbits for diffeomorphisms, Publ. I.H.E.S. 51, 137-174.
28. J. Kingman (1968) The ergodic theory of subadditive stochastic processes, J. Royal Stat. Soc. Ser. B30, 499-510.
29. S. Kotani (1982) Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrodinger operators, Proc. Kyoto Stochastic Conf.
30. J. Kurzweil (1957) Generalized ordinary differential equations and continuous dependence on a parameter, Czechoslovak Math. J. 7 (82), 415-448.
31. S.T. Liao (1966) Applications to phase-space structure of ergodic properties of the one-parameter transformation group induced on the tangent bundle by a differential system on a manifold I, (Chinese), Acta. Sci. Natur. Univ. Pekinensis 12, 1-43.
32. S.T. Liao (1973) An ergodic property theorem for a differential system, Sci. Sinica 16, 1-24.
33. M.A. Lyapunov (1892) Probleme general de la stabilite du mouvement, Ann. Math. Studies no. 17, 1947.
34. G.A. Margulis (1975) Discrete groups of motions of manifolds of non-positive curvature, in Proc. Int. Cong. Math., Vancouver, 1974, pp. 21-34.
35. Y. Matshushima (1972) Differentiable Manifolds, Marcel Dekker, New York.

36. R.K. Miller and G.R. Sell (1970) Volterra integral equations and topological dynamics, Memoir Amer. Math. Soc. No. 102.
37. V.M. Millionscikov (1968) Metric theory of linear systems of differential equations, Math. USSR-Sbornik 6, 149-158.
38. V.M. Millionscikov (1978) Typicality of almost reducible systems with almost periodic coefficients, Differential Equations 14, 448-450.
39. A.W. Naylor and G.R. Sell (1982) Linear Operator Theory in Engineering and Science, 2nd ed. Springer-Verlag, Applied Math. Sciences, Vol. 40, New York.
40. V. Nemytskii and V. Stepanov (1960) Qualitative Theory of Differential Equations, Princeton Univ. Press.
41. V.L. Novikov (1975) On almost reducible systems with almost periodic coefficients, Math. Notes 16, 1065-1071.
42. V. Oseledec (1968) A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems, Trans. Moscow Math. Soc. 19, 197-231.
43. K.J. Palmer (1980) On the reducibility of almost periodic systems of linear differential equations, J. Diff. Equations 35, 374-390.
44. K.J. Palmer (1984) Exponential dichotomies and transversal homoclinic points, J. Diff. Equations 55, 225-256.
45. L. Pastur (1980) Spectral properties of disordered systems in the one-body approximation, Comm. Math. Phys. 75, 179-196.
46. S. Pelikan (1983) The dimension of attractors are surfaces, Boston University, Ph.D. Thesis.
47. O. Perron (1930) Uber eine Matrixtransformation, Math. Zeit. 32, 465-473.
48. D. Perry (1985) A numerical study of the relationship between the measurable spectrum and the continuous spectrum, University of Minnesota Ph.D. Thesis, to appear.
49. Ya. B. Pesin (1977) Lyapunov characteristic exponents and smooth ergodic theory, Russ. Math. Surveys 32, No. 4, 55-114.
50. R. Phelps (1966) Lectures on Choquet Theory, American Book Co., New York.
51. M. Raghunathan (1979) A proof of Oseledec's multiplicative ergodic theorem, Israel J. Math. 32, 356-362.
52. D. Ruelle (1979) Ergodic theory of differentiable dynamical systems, Publ. I.H.E.S. 50, 275-320.

53. R.J. Sacker (1969) A perturbation theorem for invariant manifolds and Hölder continuity, J. Math. Mech. 18, 705-762.
54. R.J. Sacker and G.R. Sell (1974) Existence of dichotomies and invariant splittings for linear differential systems I, J. Diff. Equations 15, 429-458.
55. R.J. Sacker and G.R. Sell (1975) A spectral theory for linear almost periodic equations. Preliminary report, In International Conference on Differential Equations, pp. 698-709, Academic Press.
56. R.J. Sacker and G.R. Sell (1976a) Existence of dichotomies and invariant splittings for linear differential systems II, J. Diff. Equations 22, 478-496.
57. R.J. Sacker and G.R. Sell (1976b) Existence of dichotomies and invariant splittings for linear differential systems III, J. Diff. Equations 22, 497-522.
58. R.J. Sacker and G.R. Sell (1978) A spectral theory for linear differential systems, J. Diff. Equations 27, 320-358.
59. R.J. Sacker and G.R. Sell (1980) The spectrum of an invariant submanifold, J. Diff. Equations 37 135-160.
60. J. Selgrade (1975) Isolated invariant sets for flows on vector bundles, Trans. Amer. Math. Soc. 203, 359-390.
61. G.R. Sell (1979) Bifurcation of higher dimensional tori, Arch. Rat. Mech. Anal. 69, 199-230.
62. G.R. Sell (1985) Linearization and global dynamics, IMA Preprint 20. Proc. Internat. Congress Math., Warsaw, to appear.
63. S. Smale (1967) Differentiable dynamical systems, Bull. Amer. Math. Soc. 73, 747-817.
64. R.E. Vinograd (1956) Necessary and sufficient criteria for the behavior of solutions of regular systems, Math. Sbornik 38 (80), 23-50.
65. R.E. Vinograd, et al (1977) On the topological causes of the anomalous behavior of certain almost periodic systems (Russian), in Problems of the Asymptotic Theory of Nonlinear Oscillations, pp. 54-61, Naukova Dumka, Kiev.

Recent IMA Preprints
(continued)

34	J. Kirkwood, C.E. Wayne	Percolation in Continuous Systems
35	Luis Magalhaes	Invariant Manifolds for Functional Differential Equations Close to Ordinary Differential Equations
36	C. Eugene Wayne	The KAM Theory of Systems with Short Range Interactions II
37	Jean De Canniere	Passive Quasi-Free States of the Noninteracting Fermi Gas
38	Elias C. Aifantis	Maxwell and van der Waals Revisited
39	Elias C. Aifantis	On the Mechanics of Modulated Structures
40	William Ruckle	The Strong ϕ Topology on Symmetric Sequence Spaces
41	Charles R. Johnson	A Characterization of Borda's Rule Via Optimization
42	Hans Weinberger Kazuo Kishimoto	The Spatial Homogeneity of Stable Equilibria of Some Reaction-Diffusion Systems on Convex Domains
43	K.A. Pericak-Spector W.O. Williams	On Work and Constraints in Mixtures
44	H. Rosenberg, E. Toubiana	Some Remarks on Deformations of Minimal Surfaces
45	Stephan Pelikan	The Duration of Transients
46	V. Capasso, K.L. Cooke, M. Witten	Random Fluctuations of the Duration of Harvest
47	E. Fabes D. Stroock	The L^p -integrability of Green's functions and fundamental solutions for elliptic and parabolic equations
48	H. Brezis	Semilinear Equations in $\mathbb{R}^N$ without conditions at infinity
49	M. Slemrod	Lax-Friedrichs and the Viscosity-Capillarity Criterion
50	C. Johnson, W. Barrett	Spanning Tree Extensions of the Hadamard-Fischer Inequalities
51	Andrew Postlewaite, David Schmeidler	Revelation and Implementation under Differential Information
52	Paul Blanchard	Complex Analytic Dynamics on the Riemann Sphere
53	G. Levitt, H. Rosenberg	Topology and Differentiability of Labyrinths in the Disc and Annulus
54	G. Levitt, H. Rosenberg	Symmetry of Constant Mean Curvature Hypersurfaces in Hyperbolic Space
55	Ennio Stacchetti	Analysis of a Dynamic, Decentralized Exchange Economy
56	Henry Simpson Scott Spector	On Failure of the Complementing Condition and Nonuniqueness in Linear Elastostatics
57	Craig Tracy	Complete Integrability in Statistical Mechanics and the Yang-Baxter Equations
58	Tongren Ding	Boundedness of Solutions of Duffing's Equation
59		Abstracts for the Workshop on Price Adjustment, Quantity Adjustment, and Business Cycles
60	Rafael Rob	The Coase Theorem an Informational Perspective
61	Joseph Jerome	Approximate Newton Methods and Homotopy for Stationary Operator Equations
62	Rafael Rob	A Note on Competitive Bidding with Asymmetric Information
63	Rafael Rob	Equilibrium Price Distributions
64	William Ruckle	The Linearization Projection, Global Theories