Robustness of Estimators of the Squared Multiple Correlation and Squared Cross-Validity Coefficient to Violations of Multivariate Normality

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A monte carlo experiment was conducted to evaluate the robustness of two estimators of the population squared multiple correlation (R_p^2) and one estimator of the population squared cross-validity coefficient (R_{cv}^2) to a common violation of multivariate normality. Previous research has shown that these estimators are approximately unbiased when independent and dependent variables follow a joint multivariate normal distribution. The particular violation of multivariate normality studied here consisted of a dependent variable that may assume only a few discrete values. The discrete dependent variable was simulated by categorizing an underlying continuous variable that did satisfy the multi-

variate normality condition. Results illustrate the attenuating effects of categorization upon R_p^2 and R_{cv}^2 . In addition, the distributions of sample squared multiple correlations and sample squared cross-validity coefficients are affected by categorization mainly through the attenuations of R_p^2 and R_{cv}^2 . Consequently, the formula estimators of R_p^2 and R_{cv}^2 were found to be as accurate and unbiased with discrete dependent variables as they were with continuous dependent variables. Substantive researchers who use categorical dependent variables, perhaps obtained by rating scale judgments, can justifiably employ any of the three estimators examined here.

Multiple regression is a method used in many areas of psychology to study the relation between a set of independent variables and a dependent variable. There seem to be two distinct functions served by these studies. In research investigating theoretical relations, interest is focused on the *population* regression equation. Applied research efforts, in contrast, are usually concerned with the long run effectiveness of a particular *sample* regression equation. In neither case is the effectiveness of a sample regression equation in its derivation sample of primary interest.

The null hypothesis that all regression weights are zero in the population can be tested by a well-known procedure. Researchers frequently wish to go beyond this significance test and consider the magnitude of relation between the set of independent variables and the dependent variable. Unfortunately, the sample squared multiple correlation, R_p^2 , can provide misleading information about the magnitude of the relationship, particularly if sample size, N , is not substantially larger than the number of independent variables, k .

To eliminate the bias of R_p^2 , a number of "shrinkage" formulas have been developed. Considered here are (1) two formulas for estimating the squared multiple correlation, R_p^2 , which results when the

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population multiple regression equation is applied in the population and (2) a formula for estimating the squared cross-validity coefficient, R_{cv}^2 , which results when a *sample* multiple regression equation is applied in the population.

The first estimator of R_p^2 , usually attributed to Wherry (1931), is

$$\hat{R}_p^2 = 1 - \frac{N-1}{N-k-1} (1 - R_s^2). \quad [1]$$

This equation is appropriate for the "fixed X " or regression model (i.e., values of the independent variables are fixed or predetermined by the investigator) when an intercept parameter is included in the regression equation. The equation actually presented by Wherry (1931) is

$$\hat{R}_p^2 = 1 - \frac{N-1}{N-k} (1 - R_s^2), \quad [2]$$

which can be derived when there is no intercept parameter. Because virtually all psychologists include a constant in their regression equations, Equation 2 is not considered further. Note, however, that Equation 2 does occasionally appear in the literature (see, for example, Nunnally, 1978, p. 180).

The second formula for estimating R_p^2 examined here is the three-term expansion (Herzberg, 1969) of the Olkin-Pratt (1958) estimator,

$$\hat{R}_p^2 = 1 - \frac{N-3}{N-k-1} (1 - R_s^2) - \frac{N-3}{N-k-1} \frac{2}{N-k+1} (1 - R_s^2)^2. \quad [3]$$

This equation is appropriate under the conditions of the correlation or "random X " model (i.e., independent and dependent variables follow a multivariate normal distribution).

The precisions of Equations 1 and 3 can be evaluated (at least approximately) by analytic methods when the assumptions underlying the equations are met. For example, the Olkin-Pratt (1958) estimator is unbiased and the three-term expansion in Equation 3 of the infinite series in the Olkin-Pratt estimator is approximately unbiased.

Browne (1975) developed an estimator of R_{cv}^2 that is appropriate for the random X model. It is

$$\hat{R}_{cv}^2 = \frac{(N-k-3)\hat{R}_p^4 - \hat{R}_p^2}{(N-2k-2)\hat{R}_p^2 + k}. \quad [4]$$

Browne suggested estimating R_p^2 via Equation 1 and using

$$\hat{R}_p^4 = (\hat{R}_p^2)^2 - \frac{2k(1 - \hat{R}_p^2)^2}{(N-1)(N-k+1)} \quad [5]$$

to estimate R_p^4 .

In a recent monte carlo study, Drasgow, Dorans, and Tucker (1979) found that Equation 4 performed quite well within the scope of their simulation. It is important to note that the conditions simulated by Drasgow et al. (1979) precisely matched the assumptions of the random X model.

In sum, when used under appropriate conditions, Equations 1, 3, and 4 all appear approximately unbiased. In addition, the standard errors for Equations 1 and 3 may be approximated by analytic methods using relations presented by Kendall and Stuart (1979, pp. 362-363), and the research of Drasgow et al. (1979) can be used to examine the standard error of Equation 4.

The present research examines the effects of violating an important assumption that underlies Equations 1, 3, and 4. First studied are the effects of the violation on the sampling distributions of R_p^2

and R^2_{cv} . Clearly, if these distributions are affected in unpredictable ways, it is unlikely that Equations 1, 3, and 4 will be effective.

A Discrete Dependent Variable

The violation of multivariate normality considered here is one that occurs frequently in psychology. In particular, the dependent variable is not assumed to be continuous but, instead, is discrete. Three examples illustrate the categorical nature of important dependent variables in psychology.

The first example comes from personnel psychology. Ability tests are commonly used to predict job performance. The tests may assess cognitive, clerical, or physical abilities. It is unfortunate, but common, to see job performance assessed by a 7-point scale that asks the worker's supervisor to rate "overall job performance."

Categorical dependent variables are also encountered in research investigating consequences of workers' affective responses to their work and work role in the organization. Probably the most widely used instrument for measuring various facets of job satisfaction is Smith, Kendall, and Hulin's (1969) Job Descriptive Index (JDI). Scores on JDI scales are used to predict categorical variables such as turnover (which is dichotomous), absenteeism, and grievances filed.

Educational measurement provides a third example of a categorical dependent variable. First-year graduate grade-point average is a dependent variable frequently used in studies that investigate the validity of standardized aptitude tests such as the Graduate Record Examination. For six courses that are graded on a common 4-point scale (e.g., 1, 2, 3, and 4), the grade-point average scale contains at most 19 unique scale values, ranging from 1.0 to 4.0 in intervals of .167. In practice, however, it is common to observe severe restriction of range. If only grades of "A" and "B" are assigned to students, a dependent variable having seven or less scale values will result.

In all three of the examples described, the independent variables, although discrete, have a large number of categories. Frequency distributions are typically unimodal with gradually decreasing tails. The assumption that the independent variables follow a multivariate normal distribution probably does not do gross violence to the data. The dependent variables, in contrast, do not appear remotely normal. Only a limited number of response categories are available, and there may be restriction of range within these few categories. Consequently, the violation of multivariate normality investigated here consists of the failure of the dependent variable to have a marginal normal distribution and be jointly normal with the independent variables.

Psychometric Model

Although the dependent variables under consideration are not normally distributed, it appears reasonable to assume that there is a latent variable underlying the observed dependent variable that meets the multivariate normality assumption. The categorical dependent variable Y is therefore assumed to be formed by imposing a set of thresholds t_j or category boundaries on a continuous latent variable Y :

$$Y = \begin{cases} 1 & \text{if } Y < t_1 \\ j & \text{if } t_{j-1} \leq Y < t_j, j = 2, \dots, g-1 \\ g & \text{if } Y \geq t_{g-1}. \end{cases} \quad [6]$$

The independent variables and Y are assumed to follow a joint multivariate normal distribution. Without loss of generality, it can be assumed that Y has mean zero and unit variance.

Effects of Polychotomizing Y

A consequence of the assumption that Y and the independent variables follow a multivariate normal distribution is that Y and any individual independent variable, X_i , will follow a bivariate normal distribution. Let r_i denote the correlation between Y and X_i . Olsson, Drasgow, and Dorans (1982) have shown that if the discrete variable Y is formed from Y through Equation 6, then the correlation between X_i and Y is

$$r_d = r(X_i, Y) = q r_i. \quad [7]$$

In Equation 7 q is

$$q = \frac{1}{\sigma_Y} \sum_{j=1}^{g-1} \phi(t_j), \quad [8]$$

where σ_Y is the standard deviation of Y , and $\phi(t_j)$ is the height of the normal curve at t_j :

$$\phi(t_j) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{t_j^2}{2} \right\}. \quad [9]$$

A more general form of Equation 8 can be derived that does not constrain the scale values of the categorical criterion to be successive integers, as in Equation 6 (Olsson et al., 1982). However, the special case of successive integer scoring is studied here because most researchers use consecutive integers or some linear transformation of consecutive integers.

The scalar q is called the *categorization attenuation factor* because it provides a precise measure of the extent to which categorization reduces the linear relation between X_i and Y . It is apparent that q depends upon the number and location of the thresholds t_j . Thus, categorization attenuates r_i in the way that would be expected.

As implied by its name, the polyserial correlation is a generalization of the biserial correlation. When Y is dichotomized, it is possible to rewrite Equation 7 in the form of the relation between point biserial and biserial correlations presented by Lord and Novick (1968, p. 340). Lord and Novick illustrate the effects of dichotomizing Y by presenting a table showing q for several thresholds. For example, the maximum value of q is .798, which occurs when Y is dichotomized at its median.

Equations 7 and 8 can also be used to examine the effects of polychotomizing Y in a particular manner. Table 1 contains the eight sets of thresholds used in this study. The term "threshold vector" is used to denote a set of thresholds for categorizing Y . The values of q associated with the threshold vectors in Table 1 range from .976 to .637, and the number g of categories ranges from 2 to 11.

There are a number of interesting points that can be made from examining Table 1. First, note that with only seven well-spaced categories r_d can reach 97.6% of r_i . In other analyses (not presented here), values of q were found to be .898, .939, .960, and .997 for g equal to 3, 4, 5, and 25, respectively. However, simple expressions for the values of t_j that maximize q for a given g have not been derived. A simple procedure that yields large values of q involves placing the t_j symmetrically about 0 (i.e., the mean of Y) and spaced such that the probabilities of categories approximate the normal density function.

Table 1
Threshold Vectors and Category Probabilities

Threshold Vector	j										
	1	2	3	4	5	6	7	8	9	10	11
q = .976											
1. t_j	-1.50	-.90	-.30	.30	.90	1.50					
p_j	.07	.12	.20	.24	.20	.12	.07				
q = .798											
2. t_j	.00										
p_j	.50	.50									
3. t_j	-1.65	-1.28	-1.04	-.84	-.67	1.65					
p_j	.05	.05	.05	.05	.05	.70	.05				
4. t_j	-1.48	-1.35	-1.24	1.24	1.35	1.48					
p_j	.07	.02	.02	.78	.02	.02	.07				
5. t_j	-1.28	-1.23	-1.17	-1.13	-1.08	-1.04	-.99	-.85	-.51	.24	
p_j	.10	.01	.01	.01	.01	.01	.01	.04	.11	.29	.41
q = .637											
6. t_j	1.10										
p_j	.86	.14									
7. t_j	-1.50	-1.44	-1.38	-1.32	-1.26	-.96					
p_j	.07	.01	.01	.01	.01	.07	.83				
8. t_j	-1.82	-1.77	-1.72	-1.67	-1.62	-1.57	-1.51	1.62	1.67	1.72	
p_j	.03	.00	.00	.00	.01	.01	.01	.88	.01	.00	.04

Note: To conserve space, entries have been rounded to the second decimal. Accuracies of t_j and p_j were maintained to the fourth decimal point in all calculations.

A second conclusion evident from Table 1 is that there is *not* a unique threshold vector for each value of q . Instead, many threshold vectors can be selected to produce the same q . Table 1 also shows that threshold vectors with differing values of g can yield identical values of q .

One further observation concerning Table 1 is that increasing g does not necessarily increase q . This point is particularly important to researchers who must decide how many categories to use for their rating scales. Note that the second threshold vector, which produces a dichotomous Y , has a larger value of q than the eighth threshold vector, which yields 11 categories of Y . Table 1 illustrates that the *location* of the thresholds is very important in determining q .

Table 2 presents the skewness and kurtosis of the eight discrete variables that result from the threshold vectors in Table 1. These moment ratios are also presented for the normal, uniform, and χ^2 (one degree of freedom) distributions. The skewness of the discrete variables ranges from symmetry to about the skew of the χ^2 distribution. Kurtosis values range from flatter than the uniform distribution to almost as peaked as the χ^2 distribution. It is clear that many of these moment ratios depart substantially from the values for the normal distribution.

Table 2
Moment Ratios for Discrete Variables that Result
from Threshold Vectors Shown in Table 1

Threshold Vector	Skewness	Kurtosis
1	0.00	2.33
2	0.00	1.00
3	-1.71	4.71
4	0.00	5.73
5	-1.82	4.87
6	2.12	5.51
7	-2.78	9.20
8	-2.86	14.30
Normal	0.00	3.00
χ^2 1 df	2.83	15.00
Uniform	0.00	1.80

The design and results of a monte carlo study investigating the effects of categorization in multiple regression are described in the following section.

Method

Design of Simulation

Because R_p^2 is invariant with respect to any nonsingular linear transformation of the independent variables, any independent variable covariance matrix and vector of independent variable-dependent variable covariances that yield R_p^2 as the squared population multiple correlation may be chosen. One particularly convenient form results from (1) choosing the independent variable covariance matrix Σ_{xx} to be an identity matrix; (2) setting all elements in the vector σ_{xv} of independent variable-dependent variable covariances equal to zero except for the first element, which is set equal to R_p ; and (3) choosing a dependent variable variance σ_v^2 of unity.

The density function of sample squared multiple correlations that would result from predicting Y (if it were observable) from k independent variables in samples of size N is

$$f_{R_s^2}(p^2) = \frac{(1 - R_p^2)^{\frac{1}{2}(N-1)} (p^2)^{\frac{1}{2}(k-2)} (1 - p^2)^{\frac{1}{2}(N-k-3)}}{B(\frac{1}{2}k, \frac{1}{2}(N-k-1))} \times H(\frac{1}{2}(N-1), \frac{1}{2}(N-1), \frac{k}{2}, p^2 R_p^2), \quad [10]$$

where B is the beta function and H is the hypergeometric function (Kendall & Stuart, 1979). From Equation 10, it is clear that the particular forms of Σ_{xx} and σ_{xy} selected do not affect the sampling distribution of R_s^2 . In addition, Drasgow et al. (1979) have shown that the sampling distribution of R_{cv}^2 is also unaffected by nonsingular linear transformations. Thus, the orthogonal forms of Σ_{xx} and σ_{xy} can be used without loss of generality.

The "rectangle-tooth-tail" (Marsaglia, Ananthanarayanan, & Paul, 1976) method was used to generate pseudorandom normal numbers. The uniform numbers used in this procedure were generated by combining a multiplicative congruential generator and a shift register generator. To create data for a hypothetical individual on k predictor variables and Y , $k+1$ pseudorandom independent, standardized normal numbers were sampled. The first k numbers were taken as the scores on the independent variables and Y was obtained by

$$Y = Z_1 R_p + Z_{k+1} (1 - R_p^2)^{\frac{1}{2}}, \quad [11]$$

where Z_i is the i^{th} pseudorandom number sampled. This procedure yields samples from a $k+1$ dimensional multivariate normal distribution, where the independent variable covariance matrix is an identity matrix, Y has unit variance, and all independent variables are independent of Y except for X_1 , which has a covariance (and correlation) of R_p with Y .

Each threshold vector in Table 1 was then applied to Y , resulting in eight "observed" dependent variables. After N individuals were simulated, sample least squares regression equation and sample squared multiple correlations were computed for Y and each of the eight categorized variables. Each R_s^2 was then used to estimate R_p^2 using Equations 1 and 3. The cross-validity of each set of sample regression weights, $\hat{b}$, was computed by

$$R_{cv} = \frac{\hat{b}' \sigma_{xy}}{(\hat{b}' \hat{b})^{\frac{1}{2}} \sigma_y} \quad [12]$$

where σ_{xy} is the vector of population independent variable-discrete dependent variable covariances and σ_y is the standard deviation of the discrete dependent variable. It is obvious that the independent variable-dependent variable covariances are zero for all but X_1 , regardless of the particular threshold vector. The covariance between X_1 and a dependent variable after Y has been categorized is $q\sigma_{xy}$, where q has been calculated by Equation 8. Because R_{cv} can be negative (and this would be important, albeit unhappy, information to the applied researcher), the squared cross-validity was given the sign of R_{cv} . Finally, R_s^2 was estimated by Equation 4 for each of the nine sample regression equations, using the estimators of R_p^2 and R_p^4 suggested by Browne (1975).

To summarize briefly, samples of size N were drawn from a $k+1$ dimensional multivariate normal distribution. The first k variables were taken as the independent variables; the $(k+1)^{\text{th}}$ variable

was constructed to be a continuous dependent variable, which would *not* be observed in practice. Eight threshold vectors were applied to the continuous criterion to form categorical variables. Least squares regressions were computed for each of the nine dependent variables, and the R^2 associated with each equation was computed. Two estimates of R_p^2 , R_{cv}^2 , and an estimate of R_{cv}^2 were computed for each equation. Thus, a total of 45 squared correlations (five squared correlations for each of the nine criteria) was obtained for each sample.

Simulation Parameters

After weighing computer costs against simulating a wide variety of situations, it was decided to generate samples of size $N = 30, 60$, and 120 for populations with $k = 4$ and 12 independent variables. Values of R_p^2 for the continuous dependent variable were chosen to be .150, .236, and .370 for two reasons. First, they span much of the range of population squared multiple correlations that underlie data sets in psychological research. The second reason can be seen in Table 3: Note that when a threshold vector for which $q = .798$ is applied to a continuous dependent variable for which $R_p^2 = .370$, the categorized dependent variable has $R_p^2 = .236$. Thus, the sampling distribution of R_{cv}^2 and R_s^2 for a categorized dependent variable with $R_p^2 = .236$ can be compared to their counterparts for the continuous dependent variable with $R_p^2 = .236$. This comparison is interesting because previous research (Drasgow et al., 1979) has shown that Equations 4 and 1 produce accurate estimates of R_{cv}^2 and R_s^2 , respectively, when the variables follow a multivariate normal distribution. Similar comparisons can be made when threshold vectors for which $q = .798$ are applied to the continuous dependent variable for which $R_p^2 = .236$, and when threshold vectors with $q = .637$ are applied to the continuous dependent variable with $R_p^2 = .370$.

Table 3
Population Squared Multiple Correlations
for Continuous and Categorical Criteria

R_p^2 for Continuous Criterion	R_p^2 for Categorical Criteria With	
	$q^2 = .798^2$	$q^2 = .637^2$
.370	.236	.150
.236	.150	.096
.150	.096	.061

Results

Sampling Distribution of R_s^2

The sampling distribution of R_s^2 was first investigated for the continuous dependent variable in all 18 combinations of N , k , and R_p^2 to assess the adequacy of the random number generator. Gurland (1968) obtained a closed expression for the cumulative distribution function (cdf) of R_s^2 under the multivariate normal assumptions. It is

$$F_{R_s^2}(p^2) = \sum_{j=0}^m b_j F_{k+2j, 2m} \left(\frac{(1 - R_p^2)p^2}{1 - p^2} \right) \quad [13]$$

where

$$m = \frac{N - k - 1}{2}, \quad [14]$$

b_j is the binomial probability

$$b_j = \binom{m}{j} (R_p^2)^j (1 - R_p^2)^{m-j}, \quad [15]$$

and $F_{v,u}$ is the cumulative distribution function of a ratio χ_v^2/χ_u^2 of independent chi-square random variables with degrees of freedom v and u , respectively (Gurland, 1968, p. 279). Unfortunately, Gurland's form in Equation 13 is appropriate only if $N - k$ is an odd number. In the present research $N - k$ was even for all combinations of N and k . Although Gurland derived forms for the cdf when $(N - k)$ is even, these expressions involve infinite sums. Consequently, the cdfs for $(N - k)$ even were approximated using Equation 13 by averaging cdfs obtained from Equation 13 using $(N - 1)$ and $(N + 1)$. From other research (not presented here) it appears that the approximate cdfs have maximum errors of less than .002 for $k = 4$ and .008 for $k = 12$ within the scope of the present simulation parameters.

The sample distribution function of R_p^2 was then compared to the (approximate) theoretic distribution function for each combination of N , k , and R_p^2 for the continuous dependent variable. Kolmogorov-Smirnov significance tests were performed to assess the differences between sample and theoretic functions. Table 4 summarizes the findings. Note that for the continuous dependent variable, the actual Type 1 error rate is quite near the nominal alpha level: .055 when $\alpha = .05$, .33 when $\alpha = .25$, and .56 when $\alpha = .50$. These results provide reasonably strong support for the randomness of the pseudorandom number generator and the accuracy of the approximate cdf.¹

The Kolmogorov-Smirnov test was then utilized to compare the sampling distribution of each categorized dependent variable to the theoretic cdf function in Equation 13. In each test the population squared multiple correlation of the categorized dependent variable was the product of the squared attenuation factor (q^2) for the corresponding threshold vector and the R_p^2 of the underlying continuous variable. The results of these tests are also presented in Table 4. Again, the actual rejection rate is quite close to the nominal alpha level.² Perhaps most interesting is the actual rejection rate when $\alpha = .50$. Here the average number of rejections is 9.5, which is quite close to the expected number of 9.0. In sum, it appears that categorization had very little effect upon the sampling distribution of R_p^2 , except for the effects of changing the value of R_p^2 as specified by Equation 7.

Estimation of R_p^2

Due to the findings concerning the sampling distribution of R_p^2 described in the previous section, it is apparent that Equations 1 and 3 should estimate R_p^2 reasonably well for categorical dependent

¹It does not appear possible to compute the power of the Kolmogorov-Smirnov test. Kendall and Stuart (1979, pp. 483-484) provide approximate lower bounds for the power of the test. For example, if the largest difference between true and hypothesized cdfs is .1, if this difference occurs near the median of the two distributions, and if $\alpha = .05$, then a lower bound to the power is approximately .24. Hence, it would be expected that the null hypothesis would be rejected far more frequently than once in 18 tests.

²It should be noted that significance tests within a particular row of Table 4 are not independent. Although the threshold vectors are different, all were applied to the same continuous dependent variable. Within a given column of Table 4, however, the significance tests are independent.

Table 4
Results of Kolmogorov-Smirnov Significance Tests Comparing
Empirical and Theoretic Sampling Distributions of R^2_s

N	k	R^2_p	Continuous Criterion	Threshold Vector							
				1	2	3	4	5	6	7	8
30	4	.150	*				.11	.18	.14	*	
60	4	.150	.14	.08	*	.18		.05		.08	
120	4	.150	.13	.24	.17	*	*			*	*
30	4	.236									
60	4	.236	*	.15	.06	.05	.09	*		.06	
120	4	.236	*		*	.20	*	.16	.25	*	
30	4	.370			*	.01	.18		.02		
60	4	.370	.03	.06	*	*	.09	.04		.24	*
120	4	.370	*	.21	*	*	*	*			
30	12	.150	.21								
60	12	.150			*	*	*			*	*
120	12	.150	.08	.13	.11	.14	.19	*	.21	.25	*
30	12	.236								.09	*
60	12	.236			.18		*	.06	.09	.10	.06
120	12	.236				.04	*	*	*	.07	
30	12	.370			.18	*		*		*	
60	12	.370									
120	12	.370	.13	.16	*	.12	.04	.15	.07	*	.06
Number of Sig. Tests		18	18	18	18	18	18	18	18	18	18
Number Sig., $\alpha = .05$		1	0	0	3	3	1	2	1	0	0
Number Sig., $\alpha = .25$		6	7	5	7	7	6	6	6	6	3
Number Sig., $\alpha = .50$		10	7	12	11	11	10	10	7	11	8

Note: Each entry in the body of the table is the probability of the appropriate

Kolmogorov-Smirnov statistic. Blanks indicate probabilities greater than .5.

* $p < .50$.

variables despite the non-normality of the dependent variables. Tables 5 and 6 present mean differences between R_p^2 and estimates of R_p^2 obtained from Equations 1 and 3. In Tables 5 and 6, the columns headed by $q = 1.000$ contain mean differences based upon 100 independent replications. The columns headed $q = .798$ contain entries that are based upon 400 estimates of R_p^2 . These 400 estimates are composed of the four estimates of R_p^2 resulting from threshold vectors 2, 3, 4, and 5 across each of 100 replications. The mean differences for these threshold vectors were averaged to conserve space. Note that because the threshold vectors were applied to the same continuous dependent variable, the four estimates of R_p^2 are not statistically independent. Consequently, the standard errors of the mean differences presented in the columns headed $q = .798$ are not substantially smaller than the standard errors of the mean differences presented in the columns headed $q = 1.000$. Mean differences for threshold vectors 6, 7, and 8 were averaged for similar reasons.

Table 7 presents the root mean squared error (*RMSE*) of estimate of R_p^2 for the Wherry estimator applied to the continuous dependent variable across the 100 replications,

$$RMSE = \left\{ \frac{1}{100} \sum_{i=1}^{100} (R_p^2 - \hat{R}_{p_i}^2)^2 \right\}^{\frac{1}{2}}. \quad [16]$$

Table 7 also presents the square roots of the average *MSE* for threshold vectors with the same value of q .

It is obvious that *RMSE* is affected by the degree to which the mean estimate of R_p^2 differs from R_p^2 . However, these differences were so small in the present study that the *RMSE* is virtually identical

Table 5
Mean Differences Between R_p^2 and R_p^2 Estimated from Equation 1

R_p^2	4 Predictors			12 Predictors		
	$q = 1.000$	.798	.637	1.000	.798	.637
N = 30						
.150	.010	-.010	-.002	.019	.006	-.025
.236	-.009	-.011		.005	-.022	
.370	-.013			-.020		
N = 60						
.150	-.016	-.006	-.012	-.004	-.005	-.008
.236	-.015	-.026		-.004	-.007	
.370	-.031			-.010		
N = 120						
.150	.003	-.012	-.003	.006	-.010	.005
.236	-.015	.004		-.008	.005	
.370	.006			.010		

Note: To conserve space, means for threshold vectors with the same value of q were averaged.

Table 6
Mean Differences Between R_p^2 and R_p^2 Estimated from Equation 3

R_p^2	4 Predictors			12 Predictors		
	q = 1.000	.798	.637	1.000	.798	.637
N = 30						
.150	.025	.004	.012	.034	.021	-.011
.236	.007	.006		.021	-.006	
.370	.005			-.002		
N = 60						
.150	-.011	.000	-.006	.002	.000	-.003
.236	-.008	-.019		.003	.000	
.370	-.023			-.002		
N = 120						
.150	.005	-.010	.000	.009	-.007	.010
.236	-.012	.007		-.004	.008	
.370	.010			.014		

Note: To conserve space, means for threshold vectors with the same value of q were averaged.

Table 7
Root Mean Squared Error of Estimate for Equation 1

R_p^2	4 Predictors			12 Predictors		
	q = 1.000	.798	.637	1.000	.798	.637
N = 30						
.150	.137	.141	.141	.214	.205	.218
.236	.155	.158		.222	.224	
.370	.156			.222		
N = 60						
.150	.077	.085	.096	.113	.109	.111
.236	.091	.105		.107	.114	
.370	.114			.119		
N = 120						
.150	.054	.059	.068	.069	.076	.071
.236	.067	.071		.080	.074	
.370	.078			.078		

Note: To conserve space, MSEs for threshold vectors with the same value of q were averaged, and then the RMSE was calculated.

to the sample standard deviations of the estimator. Consequently, the *RMSE* can be (1) used to understand the accuracies of the estimators and (2) divided by 10 to yield approximate standard errors for the mean differences presented in Tables 5 and 6. Finally, a table of *RMSEs* for the Olkin-Pratt estimator is not presented because the *RMSEs* for this estimator are virtually identical to *RMSEs* seen in Table 7. The largest difference in *RMSE* between the two estimators across all cells in Table 7 is .01, which occurs in a sample of $N = 30$. For samples of $N = 120$, the *RMSEs* of the two estimators were generally within .001 of each other.

There are a number of interesting points that can be seen in Tables 5, 6, and 7. First, note that Equation 3 is virtually unbiased: R_p^2 is overestimated in 18 cells and underestimated in 14 cells. In contrast, Equation 1 has a slight tendency to underestimate R_p^2 . Here R_p^2 is overestimated in 11 cells but underestimated in 25 cells.

An important result seen in Tables 5 and 6 is that there is no apparent tendency for Equations 1 and 3 to produce an increased bias in estimating R_p^2 for the categorized dependent variable. In addition, Table 7 indicates that there is very little, if any, evidence suggesting the accuracy in estimating R_p^2 has been reduced by categorization.

Table 7 is interesting because it provides some perspective on the sample sizes required to estimate R_p^2 accurately. It is clear that using 12 independent variables in a sample of 30 observations is foolish: The *RMSE* is enormous. The most an investigator could learn would be whether R_p^2 is probably "low," "moderate," or "high." The *RMSE* is substantially reduced for 12 predictors in samples of 120 observations. In fact, the *RMSEs* here are not much larger than the *RMSEs* for 4 predictors in samples of 120 observations. This has implications for the rule of thumb stating that the ratio of N to k is the critical factor in determining the stability of results in multiple regression. Apparently the rule of thumb is overly simplistic.

Sampling Distribution of R_{cv}^2

Two-sample Kolmogorov-Smirnov significance tests were performed to assess the differences between the sampling distribution of R_{cv}^2 for continuous and categorical dependent variables. Note that R_p^2 for the categorical dependent variable (and not the R_p^2 of its underlying continuous dependent variable) was used to match a categorical dependent variable with a continuous dependent variable. Table 8 presents the results of these significance tests.

From Table 8 it is clear that the sampling distributions of R_{cv}^2 for continuous and categorical dependent variables are relatively similar. For example, the actual rejection rate is quite close to the nominal rate when $\alpha = .05$. For $\alpha = .25$, threshold vector 2 produces five rejections; however, the other threshold vectors yield frequency distributions that are significantly different from the distributions for the continuous criterion about as often as expected. Finally, when $\alpha = .50$, four of the seven threshold vectors produce one more null hypothesis rejection than expected. In sum, Table 8 does not provide compelling evidence that categorization affects the sampling distribution of R_{cv}^2 .

Estimation of R_{cv}^2

Because the continuous and categorical dependent variables yield apparently similar sampling distributions for R_p^2 and R_{cv}^2 , it appears likely that the performance of Equation 4 will not suffer when it is used with a categorical dependent variable. Tables 9 and 10 confirm this conjecture: There is no evidence to suggest that Equation 4 is less accurate after categorization. Equation 4 shows very little (if any), mean bias for either the continuous or categorical variables. In addition, *RMSEs* for the categorical dependent variables are comparable to *RMSEs* for the continuous dependent variables.

Table 8
Results of Kolmogorov-Smirnov Significance Tests Comparing Sampling
Distributions of R^2_{cv} for Continuous and Categorical Criteria

N	k	R ² _p	Threshold Vector						
			2	3	4	5	6	7	8
30	4	.150	.08	.21	*	.08	.11	.21	.11
60	4	.150			.11				*
120	4	.150		*		*		*	
30	4	.236		*		*	a	a	a
60	4	.236	.06	.08	.03	.08	a	a	a
120	4	.236	.21	.16			a	a	a
30	12	.150	.01	*	*		.08	.08	.21
60	12	.150			*	*	*		
120	12	.150		*	.11		*	*	
30	12	.236			*		a	a	a
60	12	.236	.21	.21		.21	a	a	a
120	12	.236	*		*		a	a	a
Number of Significance Tests			12	12	12	12	6	6	6
Number Significant, α = .05			1	0	1	0	0	0	0
Number Significant, α = .25			5	4	3	3	2	2	2
Number Significant, α = .50			6	8	8	6	4	4	3

Note: The entries in the body of the table are the probabilities of the two sample Kolmogorov-Smirnov statistic.

* $p < .50$.

^aNo significance test.

Discussion

Equations 1, 3, and 4 appear to be equally accurate for discrete dependent variables and continuous normally distributed dependent variables under the conditions investigated here. The results of various Kolmogorov-Smirnov tests provide the explanation for this finding: The sampling distributions of R^2 , R^2_{cv} , $\hat{R}^2_{cv}$, and the two R^2_p estimators were unaffected by categorization of the dependent variable beyond the attenuation of R^2_p . The results of this study indicate that the scope of applicability of Equations 1, 3, and 4 should be expanded to include situations where the observed dependent variable is discrete and it is reasonable to assume that there is an underlying normally distributed latent variable. Future research should be conducted to investigate the applicability of these three estimators to further violations of multivariate normality, such as categorization of the independent variables.

The robustness of most statistical methods is primarily a function of the extent to which an actual distribution differs from the assumed distribution in skewness and kurtosis. Many moment ratios of the discrete variables in Table 2 differ substantially from the corresponding normal distribution moment ratios. Differences in moment ratios larger than those seen in Table 2 could be created, but it is

Table 9
Mean Differences Between Estimated and
Actual Squared Cross-Validity

R^2_p	4 Predictors			12 Predictors		
	q = 1.000	.798	.637	1.000	.798	.637
N = 30						
.150	.002	-.015	-.005	.015	.007	.002
.236	-.002	-.013		.020	.009	
.370	-.020			.020		
N = 60						
.150	-.020	-.010	-.011	-.001	.000	-.002
.236	-.022	-.024		.002	-.003	
.370	-.033			-.006		
N = 120						
.150	.002	-.012	-.003	.005	-.008	.004
.236	-.014	.001		-.010	.002	
.370	.005			.007		

Note: To conserve space, mean differences for threshold vectors with the same value of q were averaged. Entries in Table 9 were computed as means of the differences $\hat{R}^2_{cv} - R^2_{cv}$.

Table 10
Root Mean Squared Errors of Estimate
for Equation 4

R^2_p	4 Predictors			12 Predictors		
	q = 1.000	.798	.637	1.000	.798	.637
N = 30						
.150	.132	.129	.125	.124	.120	.115
.236	.149	.155		.160	.143	
.370	.168			.164		
N = 60						
.150	.076	.082	.091	.087	.078	.080
.236	.095	.102		.092	.102	
.370	.119			.119		
N = 120						
.150	.055	.056	.066	.060	.064	.062
.236	.066	.071		.079	.071	
.370	.081			.080		

Note: To conserve space, MSEs for threshold vectors with the same value of q were averaged, and then the RMSEs were calculated.

not apparent that such artificial data sets would be representative of actual data sets seen in psychology. Consequently, more extreme violations of normality were not included in the present simulation.

This research has shown the importance of the categorization attenuation factor, q . Substantive researchers can use q to assist them in deciding how many categories should be used in the measurement of a dependent variable. The categorization attenuation factor also illustrates the importance of "anchors" or labels for rating scales. Clearly, anchors should be written such that the frequencies of selecting the various categories yield a large value of q .

In some situations a researcher may not be able to obtain a dependent variable with a large value of q (e.g., turnover data). Here an estimate of the multiple correlation obtained in predicting Y may be preferred to an estimate of the multiple correlation obtained in predicting the discrete dependent variable. Work on a closely related problem has been performed by Olsson, Drasgow, and Dorans (1982), who developed a maximum likelihood estimator of the polyserial correlation. Their estimator can be used for examining the bivariate relation between an observed continuous variable and Y . The more difficult problem of deriving an optimal estimator for the multivariate relation between a set of continuous variables and Y remains to be solved.

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