Large Sample Estimators for Standard Errors of Functions of Correlation Coefficients

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Standard errors of estimators that are functions of correlation coefficients are shown to be quite different in magnitude than standard errors of the initial correlations. A general large-sample methodology, based upon Taylor series expansions and asymptotic correlational results, is developed for the computation of such standard errors. Three exemplary analyses are conducted on a correction for attenuation, a correction for range restriction, and an indirect effect in path analysis. Derived formulae are consistent with several previously proposed estimators and provide excellent approxima-

tions to the standard errors obtained in computer simulations, even for moderate sample size ($n = 100$). It is shown that functions of correlations can be considerably more variable than product-moment correlations. Additionally, appropriate hypothesis tests are derived for these corrected coefficients and the indirect effect. It is shown that in the range restriction situation, the appropriate hypothesis test based on the corrected coefficient is asymptotically more powerful than the test utilizing the uncorrected coefficient. Bias is also discussed as a by-product of the methodology.

Many estimators in the social sciences are functions of Pearson product-moment correlation coefficients. For example, the correction of a validity coefficient for attenuation due to unreliability is a function of both the original correlation and a reliability estimate. Discussion of the corrected coefficient in its use as a parameter estimate usually revolves around its magnitude and/or practical significance. Therefore, some indication of the standard error of these corrected coefficients is crucial to the interpretability of the findings. That corrected coefficients may be greater than unity (cf. Karren, 1978) also supplies impetus to this investigation.

The present paper presents a large-sample methodology for deriving estimates of the standard errors of functions of correlation coefficients. For example, it is shown below that correlations corrected for attenuation are considerably more variable than product-moment correlations. The *APA Standards for Educational and Psychological Tests* (American Psychological Association, 1974) indicates that "where correlation coefficients are corrected for attenuation or restriction in range, full information relevant to the correction should be presented" (Section E8.2.1.). Thus, the derivation of standard errors of these corrected coefficients would represent important progress towards "full information."

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The general method for deriving approximations to standard errors of functions of correlation coefficients is based upon the union of two results: (1) the use of Taylor series expansions for derivation of asymptotic variances and (2) formulae for the asymptotic covariance *between* two correlations calculated on a common sample. This method (hereafter referred to as the delta method) will be described and then applied separately to three estimators common to the social sciences: correlations corrected for attenuation, correlations corrected for range restriction, and the estimate of an indirect effect in path analysis. The primary purpose of the proposed methodology is an enhancement of the knowledge about these functions of correlations in their use as parameter estimates and will provide support for previous empirical work. It will also be demonstrated that this methodology implies, in some cases, more powerful tests of traditional hypotheses as well as gains in the estimation of bias.

General Method

Estimates of Standard Errors

Let the estimator which is a function of Pearson product-moment correlations, r_i , be denoted by $g(r)$. (Note that the subscript i indicates a particular correlation and *not* a particular variable in the correlation.) Letting ρ_i represent the population counterpart of r_i , $\theta = \{\rho_i\}$, and $\hat{\theta} = \{r_i\}$, Kendall and Stuart (1969, p. 232) derive the following expression for the variance of $g(r)$:

$$\text{var } [g(r)] = \sum_i [g'_i(\theta)]^2 \text{var } (r_i) + \sum_{i \neq j} g'_i(\theta) g'_j(\theta) \text{cov } (r_i, r_j) + \text{(terms of order } n^{-2} \text{ or less)} \quad [1]$$

where g'_i denotes the first derivative of $g(r)$ with respect to r_i , evaluated at θ . It should be noted that if all first derivatives are zero, then further terms must be taken in the expansion.

Expressions for variances and covariances of correlation coefficients are necessary for the estimation process of Equation 1. It is well known (cf. Anderson, 1958, p. 77) that the asymptotic variance of r_i , obtained from a bivariate normal population with correlation ρ_i , is

$$\text{var } (r_i) = \frac{(1 - \rho_i^2)^2}{n} \quad [2]$$

Considerably less known are the asymptotic covariances between two correlations computed on the same sample (cf. Dunn & Clark, 1969; Kelly, 1947). The present need is for such a covariance only in situations when one variable is common to both correlations. Assuming multivariate normality, this covariance is (Kelly, 1947, p. 553)

$$\text{cov } (r_{yz}, r_{zx}) = \frac{\rho_{yz}(1 - \rho_{yx}^2 - \rho_{zx}^2) - (1/2) \rho_{yx} \rho_{zx} (1 - \rho_{yx}^2 - \rho_{yz}^2 - \rho_{zx}^2)}{n} \quad [3]$$

Then, to obtain a consistent estimate of $\text{var}[g(r)]$ from Equation 1, the terms of order n^{-2} or less can be ignored, expressions from Equations 2 and 3 can be substituted for the variance and covariance terms, and $\theta = \{\rho_i\}$ can be replaced by the sample estimate $\hat{\theta} = \{r_i\}$.

Hypothesis Testing and Bias

A corollary benefit resulting from knowledge of the standard error is the ability to test a hypothesis about the parameter of interest. Moran (1970) indicates that the hypothesis $H_0: g(\theta) = 0$ may be

tested by computing the sample estimate $g(\hat{\theta})$ divided by its estimated standard error (from Equation 1). Moran states that this ratio has, asymptotically, a standard normal distribution. In fact, since the set $\{r_i\}$ contains the maximum likelihood estimators of $\{\rho_i\}$ (cf. Kendall & Stuart, 1973, p. 306), then the test implied above is asymptotically optimal.¹

Because $g(\hat{\theta})$ is asymptotically normal, magnitudes of standard errors computed by the delta method are easily interpreted, since the relationship of the estimate to its standard error can be investigated using well-known normal density theory. Otherwise, statements about bounds on the sample estimate would be less precise (through the use of the Tchebyshev inequality, for example).

Additionally, estimation of bias is a direct consequence of the delta method. Following Kendall and Stuart (1969, p. 231) and the previous notation, bias in $g(r)$ may be expressed as

$$\text{bias} = E\{g(r)\} - g(\theta) = E\left\{\sum_i g_i'(\theta_i) (r_i - \theta_i)\right\}. \quad [4]$$

An expression for the bias in r_i , $E\{r_i - \rho_i\}$, is necessary to complete the estimation of bias in $g(r)$. Hotelling (1953, p. 212) has shown that

$$E\{r_i\} - \rho_i = (\rho_i^3 - \rho_i) (2n)^{-1} + (\text{terms of order } n^{-2} \text{ or less}). \quad [5]$$

Thus, to obtain a consistent estimate of the bias in $g(r)$, the terms of order n^{-2} or less can be ignored; Hotelling's result can be substituted into Equation 4; and $\theta = \{\rho_i\}$ can be replaced by the sample estimate $\hat{\theta} = \{r_i\}$.

Previous Methods and Results

Other strategies have been applied to estimate standard errors of functions of correlation coefficients. For example, Forsyth (1971) formed computer-generated empirical estimates of standard errors of correlations corrected for range restriction, and Gullickson and Hopkins (1976) reported empirically derived nomographs for interval estimation. Rogers (1976) investigated the utility of jack-knife procedures for estimating standard errors of correlations corrected for attenuation. These studies, however, fail to provide a closed solution to the estimation of standard errors and are dependent upon the adequacy and completeness of empirical simulations. Nonetheless, it will be demonstrated that their empirical findings are consistent with the analytic results of the delta method.

Exact solutions for standard errors have been derived for some very specific functions of correlations, namely, simple products of correlations (Bohrnstedt & Goldberger, 1969; Goodman, 1962). Unfortunately, the functions considered below are more complicated, and exact solutions have proven intractable.

Finally, estimation of standard errors by logarithmic differentials (Kelly, 1947) has been attempted on both correlations corrected for attenuation (Forsyth & Feldt, 1969) and range restriction (Kelly, 1923). In these cases, the delta method and the method of logarithmic differentials provide identical solutions. However, as the functions of correlation coefficients become more complex (e.g., products of regression weights used to estimate indirect effects in path analysis), the method of logarithmic differentials requires knowledge of complex, and usually unknown, asymptotic covariances between other functions of correlation coefficients. In contrast, the delta method requires only the re-

¹See Lord (1975) for further uses of Moran's theorem in providing the underlying basis for hypothesis tests.

sult in Equation 3. This, and the consequent direct availability of estimates of bias, suggests a preference for the delta method.

Exemplary derivations are now considered on correlations corrected for range restriction, correlations corrected for attenuation, and indirect effects in path analysis. Particular attention is given to the magnitude of the standard error, the structural form and interpretability of the formula for the standard error, and implications for hypothesis testing.

Exemplary Analyses

Correction for Attenuation

First, let the correction of a correlation, say r_{yx} , be considered for attenuation due to unreliability. Schmidt, Hunter, and Urry (1976) have argued that validity coefficients should be corrected for unreliability in the criterion only. Therefore, only corrections in one variable, say Y , are considered, although the results are easily extended to less typical corrections for attenuation (e.g., Cureton, 1965). A common estimate of the correlation between X and the true score on Y , T_y , is (Nunnally, 1967, p. 204)²

$$r_{xT_y} = \frac{r_{yx}}{(r_{yy})^{1/2}} \quad [6]$$

where r_{yy} is the estimated reliability of Y .

The derivation for the asymptotic standard error of r_{xT_y} is given in Appendix I. There are actually three formulae, depending on the method used to obtain r_{yy} : (1) r_{yy} obtained from the same sample as r_{yx} ; (2) r_{yy} obtained from an independent sample; and (3) r_{yy} given a priori by theoretical assumptions or previously accepted knowledge. In Case 1 the estimated standard error of r_{xT_y} is approximately

$$\begin{aligned} \text{S.E.}(r_{xT_y}) &= [A + B - C]^{1/2} \quad \text{where} \\ A &= n^{-1} r_{yy}^{-1} (1 - r_{yx}^2)^2, \\ B &= .25 n^{-1} r_{yy}^{-3} r_{yx}^2 (1 - r_{yy}^2)^2, \quad \text{and} \\ C &= n^{-1} r_{yy}^{-2} r_{yx} [r_{yy} (1 - 2r_{yx}^2) - .5 r_{yx}^2 (1 - 2r_{yx}^2 - r_{yy}^2)]. \end{aligned} \quad [7]$$

In Case 2, where r_{yy} is estimated independently of r_{yx} , Equation 7 gives $C = 0$, since C is proportional to the covariance of r_{yy} and r_{yx} (see Appendix I). Thus, for Case 2

$$\text{S.E.}(r_{xT_y}) = [A + B]^{1/2}. \quad [8]$$

²For recent application of this formula, see Stone (1978), who also applied a correction for range restriction to a correlation corrected for unreliability. It should be noted that the *APA Standards* (1974) states that such "chains of corrections . . . should not be seriously reported as estimates of population correlation coefficients" (Section 8.2.1).

Finally, in Case 3, where r_{yy} is known a priori, Equation 7 gives $B = 0$, since B is proportional to the variance of r_{yy} . Thus, for Case 3,

$$\text{S.E.}(r_{xTy}) = A^{1/2}. \quad [9]$$

In all three cases, application of the method of logarithmic differentials gives identical solutions. In support of the accuracy of logarithmic differential solutions, Rogers (1976, p. 131) concludes that these estimates of standard error are comparable to jackknife estimators even for sample sizes as small as 60.³

Computationally, Factor A is usually the dominating term in Equation 7. Given that $A = [\text{var}(r_{yx})]/r_{yy}$, and r_{yy} is generally less than unity, it was expected that the variance of r_{xTy} would be greater than the variance of the initial correlation, r_{yx} . Table 1 demonstrates this phenomenon by indicating that the ratio of asymptotic standard errors, $\text{S.E.}(r_{xTy})/\text{S.E.}(r_{yx})$, can be greater than unity, particularly when the reliability is relatively low. This also confirms Forsyth and Feldt's (1969) empirically based statement that "corrected correlations are considerably more variable than product-moment correlations" (p. 64).

Table 1
Ratio of Asymptotic Standard Errors of r_{xTy} to
Asymptotic Standard Errors of r_{yx} .

ρ_{yy}	ρ_{yx}
.8	.4
.8	1.0
.4	1.1
not theoretically attainable	1.5
	.3
	1.1
	1.5

Range Restriction

Suppose a bivariate distribution with a linear relationship between the variables, population correlation ρ_{yx} , and homoscedastic error is given. If the range on one variable, say X , is directly re-restricted, then, on the average, the magnitude of the obtained sample correlation, r_{yx} , will underestimate the magnitude of ρ_{yx} . A common estimate of ρ_{yx} in the unrestricted population is (cf. Ghiselli, 1964, p. 362):

$$r_{\text{range}} = \frac{r_K}{(1 - r^2 + r_K^2)^{1/2}} \quad [10]$$

³In all three exemplary analyses, the authors performed their own computer simulations. These results supported the accuracy of the delta method estimates for sample sizes as small as 100. Additionally, checks on the sampling distribution of these functions indicated no significant departure from normality for any combination of parameters tested, even at the unconservative level of $\alpha = .20$. These simulation results are available from the first author.

where r is the obtained correlation in the restricted sample and K is the ratio of the standard deviation of X in the unrestricted population (σ_x) to the standard deviation of X in the restricted population (σ'_x). Thus, $K = \sigma_x/\sigma'_x$. If the underlying distribution is bivariate standard normal, Schmidt, Hunter, and Urry (1976) have demonstrated a functional relationship between the selection ratio and K ($= \sigma_x/\sigma'_x$) such that:

$$\frac{1}{K^2} = 1 + \frac{c}{p} [f(c)] - \left[\frac{f(c)}{p} \right]^2 \quad [11]$$

where

p is the selection ratio,

c is the standardized score corresponding to the selection ratio, and

$f(c)$ is the height of the standard normal density at c .

It is shown in Appendix II that an estimate of the standard error of r_{range} is, approximately,

$$S.E. (r_{range}) = \frac{K}{[1 + r^2 (K^2 - 1)]^{3/2}} [S.E. (r)]. \quad [12]$$

Two checks on the correctness of Equation 12 are apparent: First, the expression is symmetric about $r = 0$; second, if $K = 1$, there is no change in the standard error. Some algebraic manipulation shows that Equation 12 is identical to one of several estimates of $S.E. (r_{range})$ derived by Kelly (1923) using the method of logarithmic differentials. In fact, Gullickson and Hopkins (1976) compared these estimates on the basis of empirical accuracy and concluded that the statistic equivalent to Equation 12 should be employed.

The nomographs given by Gullickson and Hopkins provide excellent support for the use of Equation 12. For example, if the sample size is $n = 100$, $r = .10$, and the selection ratio is $.40$, then Equations 10 through 12 give $K = 1.81$, $r_{range} = .30$, and an estimate of $S.E. (r_{range}) = .159$. In turn, the nomograph limits (Gullickson & Hopkins, 1976, p. 20, Figure 11) are $-.05$ and $.57$, implying an empirical standard error of $(.57 + .05)/2z_{.975} = .158$.

Indirect Effects in Path Analysis

Essentially, path analysis consists of a mapping of a set of correlations onto a causal model (cf. Kerlinger & Pedhazur, 1973; Wright, 1934). Consider the simple recursive model depicted in Figure 1, where the X_i represent random variables and the P_{ij} are labeled "path coefficients." The coefficient P_{21} is the least squares estimator obtained from the model $x_2 = P_{21}X_1 + e$, where e denotes random, homoscedastic error. Therefore, in the standardized situation considered below, $P_{21} = \rho_{12}$. Similarly, P_{32} and P_{31} are the least squares weights obtained from the model $X_3 = P_{32}X_2 + P_{31}X_1 + e$ (cf. Kerlinger & Pedhazur, 1973, p. 310).

A basic theorem of path analysis is that r_{13} can be decomposed as $r_{13} = P_{31} + P_{21}P_{32}$, where P_{31} is labeled the "direct effect of X_1 on X_3 " and $P_{21}P_{32}$ is labeled the "indirect effect of X_1 on X_3 ." That is, the indirect effect represents the route from X_1 , through X_2 , to X_3 (see Figure 1). Generally, the magnitudes of the direct and indirect effects are compared without mention of their respective standard errors. The sampling distribution of the direct effect is well known, since $\hat{P}_{31}$ is a least squares regression estimator. However, the sampling distribution of the indirect effect, even in this relatively simple model, is unknown (cf. Billings & Wroten, 1978, p. 678, for problems created by this lack of knowledge).

Figure 1
Simple Recursive Path Model Indicating
Population Path Coefficients
(Population Correlations Given in Parentheses)

Assuming standardized variables, so that $\hat{P}_{21} = r_{12}$, it is shown in Appendix III that an estimate of the asymptotic variance of the indirect effect, $\hat{P}_{21}\hat{P}_{32}$, is

$$\begin{aligned} \text{var}(\hat{P}_{21}\hat{P}_{32}) = & a^2[\text{var}(r_{12})] + b^2[\text{var}(r_{13})] + c^2[\text{var}(r_{23})] + \\ & 2ab[\text{cov}(r_{12}, r_{13})] + 2bc[\text{cov}(r_{13}, r_{23})] + \\ & 2ac[\text{cov}(r_{12}, r_{23})], \end{aligned} \quad [13]$$

where $a = (r_{12}^2 r_{23} + r_{23} - 2r_{12}r_{13})(1 - r_{12}^2)^{-2},$

$$b = -r_{12}^2(1 - r_{12}^2)^{-1}$$

and $c = r_{12}(1 - r_{12}^2)^{-1}$

Expressions for estimates of $\text{var}(r_{ij})$ and $\text{cov}(r_{ij}, r_{jk})$ in Equation 13 are obtained from Equations 2 and 3, respectively. The square root of Equation 13 provides a consistent estimate of the standard error of the indirect effect. Although Equation 13 is difficult to interpret, one aspect is quite noticeable: the appearance of $(1 - r_{12}^2)$ in the denominator of every term. Thus, if r_{12} is close to ± 1 , then the standard error may get quite large. This is consistent with regression theory, since standard errors of multiple regression weights (in this case $\hat{P}_{32}$) become large under conditions of high collinearity among the predictors (in this case r_{12} close to ± 1).

In order to demonstrate the adequacy of Equation 13, a simulation was conducted on the model in Figure 1. The population parameters are taken from a textbook chapter on path analysis (Ker-

linger & Pedhazur, 1973, p. 323). Figure 1 indicates both the path coefficients and correlations (in parentheses). The simulation drew 50 samples, each of size $n = 200$, from a standard trivariate normal distribution with intercorrelations as given in Figure 1. The sample size of 200 was chosen to approximate those of recent articles in psychology, although these articles analyze path models of greater complexity than that of Figure 1 (Grube, Greenstein, Rankin, & Kearney, 1977, $n = 218$; Tesser & Paulhus, 1976, $n = 202$).

Results of the simulation are given in Table 2. The first three rows of Table 2 contain population parameter values and their theoretical standard errors (as derived from standard regression theory), as well as their corresponding sample estimates. There is a close correspondence between population values and sample estimates. Of most interest, however, is the fourth row of Table 2, since distribution theory for the indirect effect is unknown. For the indirect effect the sample mean, .011, is close to the population value, .012. Similarly, the sample standard error of .021 confirms the utility of Equation 13 (which gives .020) as an approximation to the standard error of an indirect effect in this particular model. Indeed, as would be expected, since P_{ij} are standardized weights that are generally less than unity, the standard error of $\hat{P}_{21}P_{32}$ is substantially less than S.E. (P_{21}) or S.E. ($\hat{P}_{32}$). Regarding normality, there was no significant departure of the sampling distribution of $\hat{P}_{21}\hat{P}_{32}$ from normality, even at the unconservative significance level of $\alpha = .20$.

Hypotheses Testing

The *APA Standards* (1974, Section E8.2.1) states that validity coefficients should not be tested after they have been corrected for attenuation or range restriction. It has been demonstrated that the implied meaning of this statement is correct, i.e., sampling distributions of the corrected correlations are not identical to the sampling distribution of the uncorrected correlation. However, Moran's theorem does imply an asymptotically optimal test of the corrected coefficient, provided the appropriate change in the standard error is made through use of Equation 7 or 12. It becomes critical to assess which test is most appropriate—the test of the uncorrected validity coefficient or the Moran test of the corrected coefficient.

Table 2
Theoretical and Empirical Means and Standard Errors
From a Simple Recursive Path Model
(50 simulations with sample size 200)

Coefficient	Population Value	Average Sample Value	Theoretical Standard Error	Sample Standard Error
$P_{21}(=r_{21})$	.300	.297	.064	.065
P_{32}	.041	.036	.068	.068
$P_{31}(= \text{direct effect})$	.398	.397	.068	.057
$P_{21}P_{32}(= \text{indirect effect})$	.012	.011	.020	.021

Regarding corrections for attenuation, the large sample test for the uncorrected coefficient compares $z_1 = r_{xy}/\text{S.E.}(r_{xy})$ to a standard normal distribution. Moran's test for the correlation corrected for attenuation uses $z_2 = r_{xTy}/\text{S.E.}(r_{xTy})$. If r_{yy} is known a priori, then $B = C = 0$ in Equation 7, and $z_1 = z_2$. Thus, the two tests are asymptotically identical. If, however, r_{yy} is estimated independently of r_{yx} , B is inherently positive. In this case, r_{xTy} is still larger than r_{xy} by a factor of $r^{-1/2}_{yy}$. However, $\text{S.E.}(r_{xTy})$ has increased by a greater factor, since $\text{S.E.}(r_{xTy}) = (A + B)^{1/2} > A^{1/2} = r^{-1/2}_{yy}\text{S.E.}(r_{yx})$. Thus, $z_1 > z_2$ and it is recommended that the uncorrected coefficient be used for hypothesis testing, since it provides a more powerful test. Finally, if r_{yy} and r_{xy} are estimated from the same sample, then the ordinal relationship between $\text{S.E.}(r_{xTy}) = [A+B-C]$ and $A^{1/2} = r^{-1/2}_{yy}\text{S.E.}(r_{xy})$ is not clear because $(B - C)$ may be either positive or negative. Given that no interpretable pattern regarding the sign of $(B - C)$ has been found, it is recommended that the uncorrected coefficient be tested. Thus, if validity coefficients are to be tested in situations with correction for attenuation, the test of the uncorrected coefficient will, in general, provide a more powerful test. Of course, if the corrected coefficients are reported as point estimates, indication of their standard errors (through use of Equation 7) is still critical.

Regarding the correlation corrected for range restriction, Moran's test compares $z_3 = r_{\text{range}}/\text{S.E.}(r_{\text{range}})$ to a standard normal distribution. Again, letting $z_1 = r/\text{S.E.}(r)$, it is easily shown that $z_3/z_1 = [1 + r^2(K - 1)] > 1$ if $K > 1$. Thus, the Moran test of a validity coefficient corrected for restriction in range is asymptotically *more* powerful than the test of the uncorrected coefficient.

Kendall and Stuart (1973, p. 304) suggest that samples sizes of at least 500 may be necessary before the sampling distributions of correlations adequately approach normality. However, the empirical results of Gullickson and Hopkins (1976) indicate that tests based on z_3 provide excellent control of Type 1 error if sample sizes are as small as 100. For even smaller sample sizes, z_3 gives a liberal test of the null hypothesis, particularly when the selection ratio is low. Thus, it is recommended that, if sample sizes are moderate ($n \geq 100$), tests of validity be conducted on the *corrected* coefficients. If sample sizes are much smaller than 100, tests based on uncorrected coefficients may be more judicious.

Regarding path analytic techniques, some researchers (e.g., Spaeth, 1975) indicate that sample sizes in path analyses are usually of sufficient size so that the notion of statistical significance of direct and indirect effects is unnecessary (such notions being replaced by practical significance). A review of the psychological literature utilizing path analysis indicates otherwise. The sample sizes used by Aldrich (1972), Grube et al. (1977), Lord (1976), Simms and Szilagyi (1975), and Tesser and Paulhus (1976) were 46, 218, 144, 20, and 202, respectively. Thus, the reporting of standard errors of both direct and indirect effects is recommended. Since direct effects are estimated by regression weights, their sampling distributions are known. For indirect effects, Equation 13 provides a good first approximation to the standard error. An approximate hypothesis test is provided by Moran's theorem, although the properties of such a test under sample sizes as small as 20 or 46 need further investigation.

Bias

Approximate expressions for bias are given in the appendices. In all three instances, the bias is generally quite small and is dominated by terms of order n^{-1} . However, the functional forms of these bias estimates uncover some interesting findings.

For example, all terms of order n^{-1} in the bias of an indirect effect contain the factor $(1 - \rho^2_{12})^{-1}$. Thus, if ρ_{12} is high, the bias of the indirect effect estimate may be quite large, even though each $\hat{P}_{ij}$, being a regression weight, is unbiased. This is, of course, the same condition of high collinearity that is associated with large standard errors of the path coefficients.

Also, inspection of the bias in correlations corrected for attenuation indicates that bias could be large if r_{yy} were extremely small. This is precisely the condition that results in relatively large standard errors for r_{xTy} . These findings lend statistical support to the *APA Standards*' (1974) statement that values of r_{xTy} "are very much open to misinterpretation, especially if based on obtained coefficients that are very low" (Section E8.2). Of course, the statements in this paper are based solely on small values of r_{yy} . Contrary to any implications of the above quote, no statistical problem has been found if r_{yx} is small, at least as far as standard error and bias are concerned.

Discussion and Summary

Standard Errors

The primary purpose of the present study was the exposition of a large-sample methodology for the computation of standard errors of functions of correlation coefficients. Three exemplary statistical functions were analyzed. In general, standard errors of indirect effects in simple path analytic models are quite small. However, standard errors of correlations corrected for range restriction or attenuation may be dramatically greater than standard errors of the original correlations. For example, in *Commonwealth of Pennsylvania vs. O'Neill*^a a sample validity coefficient of .14 was corrected for range restriction due to a selection ratio of .05. Assuming bivariate normality, Equations 11 and 12 indicate that K is approximately 5 and that the corrected validity has a standard error of more than three times the standard error of the original validity coefficient.

Results from the delta method give further credence to the need for routine reporting of standard errors of functions of correlation coefficients. These standard errors enhance the correctness of discussions about the magnitudes of such estimates and represent progression towards the "full information" demand of the *APA Standards* (1974, Section E8.2.1.). Simulation studies demonstrated that the delta method provides acceptable estimates of standard errors when the sample size is moderate ($n = 100$). If n is much smaller, jackknifing is suggested (cf. Bissell & Ferguson, 1975; Rogers, 1976) as a method for estimation of standard errors, although the functional form of the equations derived from the delta method provides valuable insights, which jackknifing cannot.

Hypothesis Testing

It has been demonstrated that the delta method may lead to more powerful tests of validity coefficients. For example, in situations of range restriction, it was demonstrated that the test for the existence of validity based on the corrected coefficient is asymptotically more powerful than the test based on the uncorrected coefficient. As noted earlier, it is recommended that for situations of moderate sample size (say, 100 or greater), hypothesis tests for the existence of validity in range restriction situations be conducted on the *corrected coefficients*. This recommendation seems to contradict the *APA Standards*' (1974) statement about testing corrected coefficients. However, the APA reference point was not the test in this study, but a test of the corrected coefficient based on the *uncorrected* standard error.

The analyses of the present study clearly indicate that in situations with correlations corrected for attenuation, hypothesis tests should be conducted on the uncorrected coefficients. This is in concordance with the *APA Standards*, although, again, the APA reference point was a test of the corrected coefficient based on the uncorrected standard error.

^a*Commonwealth of Pennsylvania et al. vs. O'Neill et al.*, Civil Action No. 70-35000.

In the simple path analytic model, a large-sample test of the indirect effect has been indicated through use of Equation 13. This solves the problem (cf. Billings & Wroten, 1978, p. 678) of having to test an indirect effect "indirectly," i.e., by deleting an indirect effect if either of its component path coefficients is nonsignificant.

Functional Forms

One distinct advantage of the proposed methodology is the ability to generate explicit formulae for asymptotic standard errors and bias. Inspection of the functional form of these derived equations provides the user with potent information. For example, inspection of Equations 7 and 12 clearly indicates that standard errors of corrected coefficients may be substantially larger than standard errors of their corresponding uncorrected coefficients. This is a mathematical assertion of the intuitive belief that researchers should not test corrected coefficients for significance using traditional tests for correlations and provides support for several statements in the *APA Standards* (1974).

Additionally, inspection of expressions for bias indicates that coefficients corrected for attenuation may be quite biased (with concomitant high standard error) if reliability estimates are low. Again, this provides mathematical support to the *APA Standards'* (1974) warning about interpretation of validities corrected for attenuation when the reliability is low. Also, in the path analytic model, it was shown that estimates of indirect effects can be biased under conditions of collinearity, even though all estimates of the path coefficients, P_{ij} , are unbiased.

Finally, one extension of the methodology would involve the use of additional terms in the Taylor series expansion (see Equation 1), perhaps until some convergence criterion has been reached. However, the utility of this suggestion may be questionable (unless, of course, all first derivatives of $g(r)$ are zero), since all additional terms would be of order n^{-2} or less. More importantly, the delta method is easily extended to other functions of correlation coefficients. Given the abundance of indices and estimators in the social sciences that are functions of correlation coefficients, the delta method is a valuable tool for extending knowledge about standard errors, bias, and appropriate hypothesis tests.

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Appendix I

For the correction for attenuation estimator, following the notation of Equation 1, $\hat{\theta} = \{Q_{yx}, Q_{yy}\}$, $\hat{\theta} = \{r_{yx}, r_{yy}\}$ and $g(r) = r_{yx} r_{yy}^{-1/2}$ may be written. Taking the first derivative of $g(r)$ with respect to r_{yx} and r_{yy} and evaluating at θ gives:

$$\frac{\partial g}{\partial r_{yx}} = \rho_{yy}^{-1/2} \quad \text{and} \quad [14a]$$

$$\frac{\partial g}{\partial r_{yy}} = (-1/2) \rho_{yx} \rho_{yy}^{-3/2} \quad [14b]$$

Substitution of these results into Equation 1 and replacement of θ by $\hat{\theta}$ gives the desired approximation:

$$\begin{aligned} \text{var}[g(r)] = & r_{yy}^{-1} \text{var}(r_{yx}) + (1/4) r_{yx}^2 r_{yy}^{-3} \text{var}(r_{yy}) \\ & - r_{yx} r_{yy}^{-2} \text{cov}(r_{yx}, r_{yy}). \end{aligned} \quad [15]$$

Further substitution of Equations 2 and 3 in the above expression yields the approximation given in Equation 7.

Regarding bias, substitution of Equations 14a and 14b into Equation 4 yields:

$$E\{g(r)\} - g(\theta) = \rho_{yy}^{-1/2} E\{r_{yx} - \rho_{yx}\} - (1/2) \rho_{yx} \rho_{yy}^{-3/2} E\{r_{yy} - \rho_{yy}\} \\ + (\text{terms of order } n^{-2} \text{ or less}). \quad [16]$$

Substitution of Equation 5 for $E\{r_{ij} - \rho_{ij}\}$ and replacement of θ by $\hat{\theta}$ produces the following approximation to the bias:

$$\text{bias} = r_{yy}^{-1/2} (r_{yx}^3 - r_{yx}) (2n)^{-1} - (1/2) r_{yx} r_{yy}^{-3/2} (r_{yy}^3 - r_{yy}) (2n)^{-1} \\ = (4n)^{-1} (r_{yy})^{-1/2} (2r_{yx}^3 - r_{yx} r_{yy}^2 - r_{yx}). \quad [17]$$

Appendix II

For the range restriction estimator, following the notation of Equation 1, $\theta = \rho$ may be written, where ρ is the correlation in the restricted population, $\hat{\theta} = r_{yx}$, and $g(r) = r K (1 - r^2 + r^2 K^2)^{-1/2}$. Taking the first derivative of g with respect to r , evaluated at θ , gives:

$$\frac{\partial g}{\partial r} = K(1 - \rho^2 + \rho^2 K^2)^{-3/2} \quad [18]$$

From Equation 1, therefore,

$$\text{var}[g(r)] = K^2(1 - \rho^2 + \rho^2 K^2)^{-3} \text{var}(r) + \\ (\text{terms of order } n^{-2} \text{ or less}). \quad [19]$$

Dropping the terms of order n^{-2} or less, taking the square root of both sides, and replacing θ by $\hat{\theta}$ produces Equation 12.

Regarding bias, substitution of Equations 18 and 5 into Equation 4 yields:

$$\text{bias} = K(1 - \rho^2 + \rho^2 K^2)^{-3/2} (\rho^3 - \rho) (2n)^{-1} + \\ (\text{terms of order } n^{-2} \text{ or less}). \quad [20]$$

Note that if $K > 1$ and $\rho > 0$, then $(1 - \rho^2 + \rho^2 K^2) > 0$ and $(\rho^3 - \rho) < 0$, so that the bias is generally negative.

Appendix III

If the variables are standardized in the estimation of an indirect effect, then $\hat{P}_{21} = r_{12}$ and $\hat{P}_{32} = (r_{23} - r_{13}r_{12})(1 - r_{12}^2)^{-1}$. Following the notation of Equation 1, $\theta = \{\rho_{12}, \rho_{13}, \rho_{23}\}$, $\hat{\theta} = \{r_{12}, r_{13}, r_{23}\}$, and $g(r) = \hat{P}_{21}\hat{P}_{32} = r_{12}(r_{23} - r_{13}r_{12})(1 - r_{12}^2)^{-1}$. Taking the first derivative of $g(r)$ with respect to r_{12} , r_{13} , and r_{23} and evaluating at θ gives:

$$a(\theta) = \frac{\partial g}{\partial r_{12}} = (\rho_{12}^2 \rho_{23} + \rho_{23} - 2\rho_{12}\rho_{13})(1-\rho_{12}^2)^{-2}, \quad [21a]$$

$$b(\theta) = \frac{\partial g}{\partial r_{13}} = -\rho_{12}^2 (1-\rho_{12}^2)^{-1}, \quad \text{and} \quad [21b]$$

$$c(\theta) = \frac{\partial g}{\partial r_{23}} = \rho_{12} (1-\rho_{12}^2)^{-1}. \quad [21c]$$

Substitution of Equations 21a, 21b, and 21c into Equation 1 and replacement of θ by $\hat{\theta}$ gives the desired approximation:

$$\begin{aligned} \text{var}[g(r)] = & [a(\hat{\theta})]^2 \text{var}(r_{12}) + [b(\hat{\theta})]^2 \text{var}(r_{13}) + [c(\hat{\theta})]^2 \text{var}(r_{23}) \\ & + 2 a(\hat{\theta}) b(\hat{\theta}) \text{cov}(r_{12}, r_{13}) + 2 b(\hat{\theta}) c(\hat{\theta}) \text{cov}(r_{13}, r_{23}) \\ & + 2 a(\hat{\theta}) c(\hat{\theta}) \text{cov}(r_{12}, r_{23}). \end{aligned} \quad [22]$$

Regarding bias, substitution of Equations 21a, 21b, and 21c into Equation 4 yields:

$$\begin{aligned} E\{g(r)\} - g(\theta) = & a(\theta) E\{r_{12}^{-\rho_{12}}\} + b(\theta) E\{r_{13}^{-\rho_{13}}\} \\ & + c(\theta) E\{r_{23}^{-\rho_{23}}\} \\ & + (\text{terms of order } n^{-2} \text{ or less}). \end{aligned} \quad [23]$$

Substitution of Equation 5 for $E\{r_{ij}^{-\rho_{ij}}\}$ and replacement of θ by $\hat{\theta}$ yields the following approximation to the bias:

$$\begin{aligned} \text{bias} = & (1-r_{12}^2)^{-1} (2n)^{-1} [(r_{12}^2 r_{23} + r_{23}^{-2} r_{12} r_{13})(1-r_{12}^2)^{-1} (r_{12}^3 - r_{12}) \\ & - r_{12}^2 (r_{13}^3 - r_{13}) + r_{12} (r_{23}^3 - r_{23})]. \end{aligned} \quad [24]$$

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